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Singular limits of reaction diffusion equations of KPP type in an infinite cylinder

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Dedicated to my family for all their support during these years.

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Singular limits of reaction diffusion equations of KPP type in an infinite cylinder

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In this thesis, we establish the asymptotic analysis of the singularly perturbed reaction diffusion equation

$$u_t - \epsilon Tr[AD_x^2 u] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 u] + \frac{1}{\epsilon} f(x, y, u) = 0$$

for $(x, y, t) \in \mathbb{R}^n \times \Omega \times (0, T)$. Here $\Omega \subset \mathbb{R}^m$ is a bounded convex set with smooth boundary and $\delta = \delta(\epsilon)$ with $\delta \to 0$ as $\epsilon \to 0^+$. The reaction term f is of KPP type, i.e., f has two equilibrium states: $u = 1$ is stable and $u = 0$ is unstable. Our goal is to characterize the limiting behavior of the solutions as $\epsilon \to 0^+$ using viscosity solutions methods. Our results establish the specific dependency on the coefficients of this equation and the size of the parameter δ with respect to ϵ. The analyses include the equation subject to Dirichlet and Neumann boundary conditions. In both cases, the solutions u^ϵ converge locally uniformly to the equillibria of the reaction term f. We characterize the limiting behavior of the solutions through the viscosity solution of a variational

inequality. To construct the coefficients defining the variational inequality, we apply concepts developed for the homogenization of elliptic operators. In chapter two, we derive the convergence results in the Neumann case. The third chapter is dedicated to the analysis of the Dirichlet case.

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Chapter 1

Preliminaries

1.1 Introduction

The reaction diffusion equations of KPP type

$$u_t - \Delta u + f(u) = 0 \qquad \text{in} \quad \mathbb{R} \times (0, T)$$
$$u(x, 0) = \mathbf{1}_{(-\infty, 0]} \qquad \text{in} \quad \mathbb{R}$$

were introduced by Kolmogorov-Petrovsky-Piskhunov [1] and Fisher [14] to study the expansion of advantageous genes. Later this kind of equations appeared modelling phenomena from physics, biology and chemical kinetics [8, 12, 19, 20].

The reaction term $f(u) = u(1 - u)$ provides this equation with two equilibria $u = 0$ (unstable) and $u = 1$ (stable). It was shown that the long time behavior of the solutions tend to the shape of the travelling wave solutions with velocity c connecting both equilibria. Hence, the solutions of the KPP equations posses an asymptotical speed of propagation defined through the speed of their travelling wave solutions. The properties of travelling waves for KPP equations have been studied extensively since the early 1900's (see [21] and references therein).

To investigate the existence of asymptotic speeds for KPP equations in nonhomogeneous media, Freidlin [3] considered equations with slowly changing media. For small parameter ϵ, he proposed the following problem

$$u_t - Tr[A(\epsilon x)D_x^2 u] + f(\epsilon x, u) = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R} \times (0,\infty)$$

For $\epsilon > 0$ sufficiently small, the coefficients of this equation vary slowly. Let the function u^ϵ be defined by $u^\epsilon(x,t) = u(\frac{x}{\epsilon}, \frac{t}{\epsilon})$. Then u^ϵ satisfies

$$u_t - \epsilon Tr[AD_x^2 u] + \frac{1}{\epsilon} f(x,u) = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{1.1.1}$$

The question addressed by Freidlin for equation (1.1.1) is: what is the behavior of u^ϵ as $\epsilon \to 0$? Using large deviation techniques, Freidlin [3, 4] established that u^ϵ converges to a piecewise constant function attaining the two equilibria of this equation. He characterized the regions of convergence to each equilibria as the zero and positive set of a nonnegative action functional $J(x,t)$.

A different approach to analyze the asymptotics of (1.1.1) was introduced by Evans and Souganidis [15]. They analyzed the asymptotic behavior of the solutions using only PDE techniques. In [15], u^ϵ is shown to have the following formal expansion

$$u^\epsilon(x,t) = e^{-\frac{I^\epsilon(x,t)}{\epsilon}} \qquad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,T).$$

It is established that $I^\epsilon \to I$ locally uniformly in $\mathbb{R}^n \times (0,T)$ and the function $I(x,t)$ is characterized, using the stability property of viscosity solutions, as the solution of a variational inequality. The function I determines the regions where u^ϵ converge to each of the equilibria in the following way: $u^\epsilon \to 0$ locally uniformly on $\{I > 0\}$ and $u^\epsilon \to 1$ locally uniformly on Int $\{I = 0\}$.

In a later paper, Friedlin [4] considers, for ϵ and $\delta > 0$, the asymptotic behavior of the solutions $u^{\epsilon,\delta}$ of

$$u_t - \epsilon Tr[AD_x^2 u] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 u] + \frac{1}{\epsilon} f(u) = 0 \qquad (1.1.2)$$

in a smooth infinite cylinder $C = \{(x, y, t) \in \mathbb{R} \times \Omega \times (0, T)\}$, where Ω is bounded, and $\delta = \delta(\epsilon)$ with $\delta \to 0^+$ as $\epsilon \to 0^+$. His analysis include Neumann and Dirichlet boundary conditions. In each case, Freidlin established the existence of a continuous nonnegative function $J(x, t)$ with the property that, in the Neumann case, $u^\epsilon \to 0$ on $\{J > 0\} \times \overline{\Omega}$ and $u^\epsilon \to 1$ in Int $\{J = 0\} \times \overline{\Omega}$. In the case of Dirichlet boundary conditions [3], with $\delta = 1$ and homogeneous coefficients, he showed that $u^\epsilon \to \varphi(y)$ in Int $\{J = 0\} \times \Omega$ where φ satisfies

$$\begin{aligned} -Tr[BD^2\varphi] + f(\varphi) &= 0 \quad \text{in} \quad \Omega \\ \varphi &= 0 \quad \text{in} \quad \partial\Omega \\ 0 < \varphi &\leq 1 \quad \text{in} \quad \Omega \end{aligned}$$

and $u^\epsilon \to 0$ on $\{J > 0\} \times \Omega$.

In this thesis, we establish the asymptotic behavior of u^ϵ, the solution of (1.1.2), using only PDE techniques. We restrict our attention to the cases where $\delta = \epsilon^\alpha$ and $\alpha \in (0, 1)$. The case $\alpha \geq 1$ follows from similar arguments as in [15]. It is rigourously established that u^ϵ has a formal asymptotic expansion

$$u^{\epsilon,\delta}(x, y, t) = e^{-\frac{1}{\epsilon} v^\epsilon(x,y,t)}.$$

The function v^ϵ satisfies an equation of the following type:

$$v_t - \epsilon Tr[AD_x^2 v] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 v] + Tr[AD_x v \otimes D_x v] + \frac{\delta^2}{\epsilon^2} Tr[BD_y v \otimes D_y v] + c(x,y) - be^{-\frac{v}{\epsilon}} = 0 \qquad (1.1.3)$$

for some $b > 0$ and $c(x,y)$ to be defined in terms of the nonlinearity f. Our choice for δ provides equation (1.1.3) with unbounded coefficients as $\epsilon \to 0^+$. This fact requires us to introduce ideas from homogenization of elliptic operators to analyze the effects of the unbounded terms in this equation as $\epsilon \to 0$.

Homogenization of differential operators deals with the analysis of processes which take place in heterogenous media such as composite materials. In such media there are two length scales: a microscopic one and a macroscopic one. The mathematical models describing the phenomena in heterogenous media are given by microscopic laws that involve a small parameter $\epsilon > 0$. This small parameter represents the ratio between the two scales. Homogenization theory is interested in deriving macroscopic approximations which consider the local effects. Such macroscopic effects are derived by establishing the asymptotics of the model as the small parameter ϵ tends to zero. Homogenization problems have been studied extensively (see for example [18] and references therein). Homogenization techniques involving viscosity solutions, were developed in [13] for first order operators and for second order elliptic operators in [2]. The main tool is to study an associated equation called the cell problem. The cell problem provides the law for the limiting macroscopic effects for the phenomena under study.

Our formal asymptotic expansion for v^ϵ, as defined in (1.1), is

$$v^\epsilon(x,y,t) = V(x,t) + \frac{\delta}{\epsilon}\chi^\delta(y),$$

where χ^δ is the solution of an appropriate cell problem. The above expansion for v^ϵ implies, in every case, that $v^\epsilon(x,y,t)$ converges locally uniformly to a continuous function $V(x,t)$ since $\frac{\delta}{\epsilon} \to 0$ as $\epsilon \to 0$. The limit function V describes the sets where the collection of functions u^ϵ converge locally uniformly to the equilibria of equation (1.1.2).

Previous research involving singularly perturbed equations includes the analysis from Bardi et al. [11] on Hamilton-Jacobi-Bellman-Isaacs equations. They considered u^ϵ as the solution of

$$\begin{aligned} u_t + H(x,y,D_x u, \frac{1}{\epsilon}D_y v, D_x^2 u, \frac{1}{\epsilon}D_y^2 u) = 0 \qquad &\text{for} \quad (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^m \times (0,T) \\ u(x,y,0) = h(x,y) \end{aligned}$$

where

$$H(x,y,p,q,X,Y) = \max_{\alpha\in A}\min_{\beta\in B}\{L^{\alpha,\beta}(x,y,p,q,X,Y)\},$$

$L^{\alpha,\beta}(x,y,p,q,X,Y)$ is a collection of linear operators periodic in the y variable, and A and B are compact control sets. In [11], a general analysis of the asymptotics of such equations is developed. Under certain assumptions on $L^{\alpha,\beta}$, $u^\epsilon \to \overline{u}(x,t)$ locally uniformly. The function $\overline{u}$ satisfies a second order parabolic equation with finite initial data $\overline{h}$. The quadratic growth in the gradient in (1.1.3) introduces the following crucial technical issue not present in [11]. In order to establish the convergence of the solutions to (1.1.2), new bounds for v^ϵ and its gradient, uniform in ϵ, must be provided. Due to the diverging terms in the derivatives of the transversal variable y in (1.1.3), the gradient bounds obtained in [15] can not be extended to our case. The gradient bounds are obtained for each boundary condition. The role of the gradient bounds in this case are necessary to provide the equations involved in our analysis with comparison principles.

The organization of this thesis is the following: in Chapter 1 we introduce the mathematical background required through our analysis. A description of our assumptions and results are included. Chapter 2 and 3 deal with the asymptotic behavior of (1.1.2) under Neumann and Dirichlet boundary conditions, respectively. In each case, the first step is to obtain uniform in ϵ bounds in $W^{1,\infty}_{loc}$ for the solutions of (1.1.3). The variational inequality satisfied by V is established through the analysis of the appropriate cell problem. The properties of the cell problems are discussed under each boundary condition. These two steps provide us with local uniform limits of the solutions to (1.1.3). Through this limit function, the asymptotic behavior of the solutions to (1.1.2) is established. The appendix is devoted to provide gradient bounds used for the cell problem in each case.

1.2 Mathematical background

1.2.1 Notation

We start by introducing the notation used throughout the manuscript.

- Int Ω is the set of interior points of $\Omega \subset \mathbb{R}^n$.
- $\overline{\Omega}$ denotes the closure of the set Ω.
- $K \subset\subset \Omega$ if $\overline{K}$ is a compact subset of Ω.
- $B_r(x)$ is the open ball of radius r with center in x.
- $Tr[A]$ is the trace of a matrix A.
- S^n is the set of $n \times n$ symmetric matrices.
- supp f denotes the support of a real valued function f.
- $\mathbf{1}_\Omega$ is the characteristic function of the set Ω.
- For $0 < t < T$ define $Q_{t,T} = \mathbb{R}^n \times \Omega \times (t, T)$.
- $Q_T = Q_{0,T}$.
- $\partial Q_T = \mathbb{R}^n \times \partial\Omega \times (0, T)$.
- For $\Omega \subset \mathbb{R}^n$ bounded define $|\Omega|$ as the diameter of Ω.
- For a function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ define $f^+(x) = \max[f(x), 0]$ and $f^-(x) = \min[f(x), 0]$.

1.2.2 Viscosity solutions

In this section the notion of viscosity solutions used in this manuscript is discussed. The discussion does not attempt to present this topic in full generality. Only viscosity solutions for continuous operators are presented. For an extensive exposition on viscosity solutions, the User's Guide [7] is recommended.

To state the notion of viscosity solutions, consider the following equation

$$F(D^2u, Du, u, x) = 0 \qquad \text{in} \quad \Omega, \tag{1.2.1}$$

where $\Omega \subset \mathbb{R}^n$ is open. The assumptions on F are:

A1) $F(X, p, r, x)$ is continuous in $S^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega$.

A2) F is degenerate elliptic i.e. for $X \leq Y$ then $\forall (p, r, x) \in \mathbb{R}^n \times \mathbb{R} \times \Omega$, $F(X, p, r, x) \geq F(Y, p, r, x)$.

A3) $\forall (X, p, x) \in S^n \times \mathbb{R}^n \times \mathbb{R}^n$, $F(X, p, \cdot, x)$ is increasing in r.

For any function $u : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ define u^* and u_* to be the upper and lower envelope of u to be defined by

$$\begin{cases} u^*(x) = \lim_{r\to 0} \sup\{u(y) | y \in \Omega, |x - y| < r\} \\ u_*(x) = \lim_{r\to 0} \inf\{u(y) | y \in \Omega, |x - y| < r\} \end{cases}$$

The upper envelope u^* of a function u is the lowest upper continuous function such that $u \leq u^*$. Similarly, the lower envelope u_* is the largest lower continuous function such that $u \geq u_*$.

Definition 1.2.1. A function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of (1.2.1) if

$$F(D^2\phi(x_0), D\phi(x_0), u^*(x_0), x_0) \leq 0.$$

whenever $\phi : \Omega \to \mathbb{R}$ is smooth and $u^* - \phi$ has a local maximum at $x_0 \in \Omega$. Similarly, u is a viscosity supersolution of (1.2.1) if

$$F(D^2\phi(x_0), D\phi(x_0), u_*(x_0), x_0) \geq 0.$$

whenever $\phi : \Omega \to \mathbb{R}$ is smooth and $u_* - \phi$ has a local minimum at $x_0 \in \Omega$.

In the case of parabolic equations

$$u_t + F(D^2u, Du, u, x, t) = 0 \qquad \text{in} \quad \Omega \times (0, T), \tag{1.2.2}$$

assume that the function F satisfies properties $A1$) and $A2$). Assumption $A3$) can be weakened for parabolic equations as follows:

A3') There exist a constant $K \geq 0$ such that $\forall (X, p, x, t) \in S^n \times \mathbb{R}^n \times \Omega \times \mathbb{R}$, $F(X, p, r, x, t) + Kr$ is increasing in r.

Definition 1.2.2. A function $u : \Omega \times (0, T) \to \mathbb{R}$ is a viscosity subsolution of (1.2.2) if

$$\phi_t + F(D^2\phi, D\phi, u^*, x, t) \leq 0 \qquad \text{at} \quad (x_0, t_0)$$

whenever $\phi : \Omega \times [0, T] \to \mathbb{R}$ smooth and $u^* - \phi$ has a local maximum at $(x_0, t_0) \in \Omega \times (0, T)$. Similarly, a function $u : \Omega \times (0, T) \to \mathbb{R}$ is a viscosity supersolution of (1.2.2) if

$$\phi_t + F(D^2\phi, D\phi, u_*, x, t) \geq 0 \qquad \text{at} \quad (x_0, t_0)$$

whenever $\phi : \Omega \times [0, T] \to \mathbb{R}$ is smooth and $u_* - \phi$ has a local minimum at $(x_0, t_0) \in \Omega \times (0, T)$.

The definition of viscosity solutions can be extended to include boundary value problems

$$F(D^2u, Du, u, x) = 0 \qquad \text{in} \quad \Omega \tag{1.2.3}$$

$$B(Du, u, x) = 0 \qquad \text{in} \quad \partial\Omega. \tag{1.2.4}$$

The boundary of Ω is assumed to be smooth. The boundary operators considered are

$$\begin{aligned} B(p, r, x) &= \langle p, n(x) \rangle && \text{Neumann B.C.} \\ B(p, r, x) &= r - \psi(x) && \text{Dirichlet B.C.} \\ B(p, r, x) &= \begin{cases} \langle p, n(x) \rangle & x \in \Gamma_1 \\ r - \psi(x) & x \in \Gamma_2 \end{cases} && \text{Mixed B.C.} \end{aligned}$$

where $n(x)$ is the unit outward normal vector field at $\partial\Omega$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_2 is open.

Definition 1.2.3. A function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of (1.2.3) with boundary conditions (1.2.4), if

$$\min \left[F(D^2\phi, D\phi, u^*, x_0), B(D\phi, u^*, x_0) \right] \leq 0 \qquad \text{if} \quad x_0 \in \partial\Omega$$

$$F(D^2\phi, D\phi, u^*, x_0) \leq 0 \qquad \text{if} \quad x_0 \in \Omega$$

whenever $\phi : \overline{\Omega} \to \mathbb{R}$ smooth and $u^* - \phi$ has a local maximum at $x_0 \in \overline{\Omega}$. Similarly, u is a viscosity supersolution of (1.2.3) if

$$\max \left[F(D^2\phi, D\phi, u_*, x_0), B(D\phi, u^*, x_0) \right] \geq 0 \qquad \text{if} \quad x_0 \in \partial\Omega$$

$$F(D^2\phi, D\phi, u_*, x_0) \geq 0 \qquad \text{if} \quad x_0 \in \Omega$$

whenever $\phi : \overline{\Omega} \to \mathbb{R}$ smooth and $u^* - \phi$ has a local minimum at $x_0 \in \overline{\Omega}$.

In all the above cases, a function u is a viscosity solution if it is both a subsolution and a supersolution. It is important to mention that an equivalent definition of viscosity subsolution and supersolutions can be formulated replacing the terms local maximum and local minimum for strict local maximum and strict local minimum.

The main ingredient to obtain a uniqueness results is comparison principles. A comparison principle is a statement of the following type: if u and v are a subsolution and a supersolution satisfying

$$0 = \sup_{x \in \partial\Omega} [u^* - v_*] \tag{1.2.5}$$

then $u \leq v$ on $\overline{\Omega}$. Similar version can be formulated in the parabolic case replacing $\partial\Omega$ by $\partial\Omega \times [0, T] \cup \Omega \times \{0\}$ in (1.2.5).

The existence of viscosity solutions is obtained via Perron's Method. To use this method, it is necessary to construct a subsolution u and a supersolution v that satisfy the boundary conditions at the boundary. Then comparison principle would imply the existence of the unique viscosity solution to the problem satisfying the boundary conditions.

In the special case of viscosity solutions of variational inequalities

$$\begin{aligned} &\min[u_t + H(x, Du), u] = 0 && \text{in } \mathbb{R}^n \times (0, T) \\ &u = h && \text{in } \mathbb{R}^n \end{aligned} \tag{1.2.6}$$

where the hamiltonian $H(x, p)$ is assumed to satisfy:

H1) For every $x \in \mathbb{R}^n$, $p \to H(x, p)$ is convex.

H2) For each $R > 0$, there exists a constant $C_R > 0$ such that for all x, $y \in \mathbb{R}^n$ and $p, q \in B_R(0)$

$$|H(x,p) - H(x,q)| \leq C_R|p-q|(|p| + |q|)$$

holds and

$$|H(x,p) - H(y,p)| \leq C_R|x-y||p|^2$$

H3) There exist constants $A \geq a > 0$ and $B \geq b > 0$, such that for all $x, p \in \mathbb{R}^n$

$$a|p|^2 - b \leq H(x,p) \leq A|p|^2 + B.$$

Definition 1.2.4. A continuous function $u : \mathbb{R}^n \times [0,T] \to [0,\infty)$ is a viscosity subsolution of (1.2.6) if for all $\phi : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ smooth, if $u - \phi$ has a local maximum at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$ then

$$u_t + H(x, Du) \geq 0 \qquad \text{at} \quad (x_0, t_0)$$

and, if for all $\phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ smooth, if $u - \phi$ has a local minimum at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$ with $u(x_0,t_0) > 0$ then

$$u_t + H(x, Du) \leq 0 \qquad \text{at} \quad (x_0, t_0).$$

The existence and uniqueness of viscosity solutions of this type of equation is established in [15].

Next, we give a representation formula for the unique viscosity solutions of variational inequalities like (1.2.6). Let

$$L(x,q) = \sup_{p \in \mathbb{R}^n} \{q \cdot p - H(x,p)\}$$

be the Lagrangian associated to the hamiltonian H. Fix $T > 0$ and for $0 \leq t \leq T$ define the space $L^2(t) = L^2((t,T), \mathbb{R}^n)$. An element $z(\cdot) \in L^2(t)$ is called a control function.

A stopping time after t is a function $\gamma : L^2(t) \to [t,T]$, such that for all $s \in [t,T]$, and $z, \bar{z} \in L^2(t)$, if $z(\tau) = \bar{z}(\tau)$ for almost every $\tau \in [t,s]$ and $\gamma[z(\cdot)] \leq s$, then $\gamma[\bar{z}(\cdot)] = \gamma[z(\cdot)]$. Let denote by $\Gamma(t)$ be the set of all stopping times after t.

Then, for each $x \in \mathbb{R}^n$ and $0 \leq t < t + \tau \leq T$

$$V(x,t) = \sup_{\gamma \in \Gamma(t)} \inf_{z(\cdot) \in N(t)} \Bigg[\int_t^{\min[t+\tau, \gamma[z(\cdot)]]} L[x(s), -z(s)] ds \qquad (1.2.7)$$
$$+ \mathbf{1}_{\{\gamma[z(\cdot)] \geq t+\tau\}} V(x(t+\tau), t+\tau) \Bigg]$$

where

$$x(s) \equiv x + \int_t^s z(\tau) d\tau \qquad (t \leq s \leq T)$$

Then $V(x,t)$ is a Lipschitz continuous viscosity solution of

$$\max[V_t + H(x, DV), V] = 0 \qquad \text{in } \mathbb{R}^n \times (0,T).$$

For a proof of this fact, see appendix in [15].

1.3 The convergence results

Let $\Omega \subset \mathbb{R}^m$ be a convex bounded set with sufficiently smooth boundary. The parameter $\delta > 0$ is given by $\delta = \epsilon^\alpha$ with $\epsilon > 0$ and $\alpha \in (0,1)$. Let

$u^\epsilon(x, y, t)$ be the solution of

$$u_t - \epsilon Tr[AD_x^2 u] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 u] + \frac{1}{\epsilon} f(x, y, u) = 0 \quad \text{in} \quad \mathbb{R}^n \times \Omega \times (0, T)$$
$$u(x, y, 0) = u_0(x, y) \quad \text{in} \quad \mathbb{R}^n \times \Omega \tag{1.3.1}$$

where u_0 is a smooth function satisfying

- $0 \leq u_0 \leq 1$ in $\mathbb{R}^n \times \overline{\Omega}$.
- supp $u_0 \subset\subset \mathbb{R}^n \times \Omega$.

Two type of boundary conditions in $\mathbb{R}^n \times \partial\Omega \times (0, T)$ are considered

$$u = 0 \qquad \text{Drichlet}$$
$$\langle D_y u, n \rangle = 0 \qquad \text{Neumann}$$

where $(0, n(y))$ is the unit outward normal vector at $\partial\Omega$. The nonlinearity f is smooth of KPP type, i.e. for each $(x, y) \in \mathbb{R}^{n+m}$

$$\begin{cases} f(x, y, u) > 0 & \text{for } u \in (0, 1) \\ f(x, y, u) < 0 & \text{for } u \in (-\infty, 0) \cup (1, \infty) \end{cases} \tag{1.3.2}$$

and

$$\frac{\partial f}{\partial u}(x, y, 0) = \sup_{0 \leq u \leq 1} \frac{f(x, y, u)}{u} = \sup_{0 \leq u \leq 1} \overline{c}(x, y, u) = c(x, y), \tag{1.3.3}$$

where $c(x, y)$ is smooth, bounded and Lipschitz continuous.

The matrices A and B are symmetric, smooth, bounded, Lipschitz, positive definite. There exist two positive constants μ and Λ such that

$$\mu I \leq A, B \leq \Lambda I \qquad \text{for} \quad (x, y) \in \mathbb{R}^n \times \overline{\Omega}.$$

Under the following change of variable

$$v^\epsilon = -\epsilon \log u^\epsilon, \tag{1.3.4}$$

then v^ϵ satisfy

$$\begin{aligned} v_t - \epsilon Tr[AD_x^2 v] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 v] + Tr[AD_x v \otimes D_x v] + \frac{\delta^2}{\epsilon^2} Tr[BD_y v \otimes D_y v] \\ + \overline{c}(x, y, e^{-\frac{v}{\epsilon}}) = 0 \end{aligned} \tag{1.3.5}$$

$$v(x,y,0) = \begin{cases} -\epsilon \log u_0(x,y) & (x,y) \in \text{Int}\{\text{ supp g }\} \\ \infty & \text{otherwise.} \end{cases}$$

The boundary conditions for v^ϵ on $\mathbb{R}^n \times \partial\Omega \times (0,T)$ are

$$v^\epsilon = \infty \qquad \text{(Drichlet case)} \tag{1.3.6}$$

$$\langle D_y v, n \rangle = 0 \qquad \text{(Neumann case).} \tag{1.3.7}$$

In the case of the Neumann boundary condition, it is shown that there exist a Lipschitz continuous function $V(x,t)$ such that $v^\epsilon \to V(x,t)$ locally uniformly in $\mathbb{R}^n \times \overline{\Omega} \times (0,T)$. For the case of infinite Drichlet boundary conditions, the convergence of v^ϵ to V is locally uniformly in $\mathbb{R}^n \times \Omega \times (0,T)$. The convergence of v^ϵ is established by obtaining uniform in ϵ gradient bounds. The function $V = V(x,t)$ in each case (1.3.6), (1.3.7) is shown to be viscosity solution of the variational inequality

$$\min[V_t + H(x, DV), V] = 0 \qquad \text{in } \mathbb{R}^n \times (0,T)$$
$$\begin{cases} 0 & x \in \text{Int}\{x | \text{ there exist y s.t. } \quad u_0(x,y) > 0\} \\ \infty & x \in \overline{\{x | \text{ there exist y s.t. } \quad u_0(x,y) > 0\}}^c \end{cases}$$

where $H(x,p) = \lim_{\delta\to 0} H^\delta(x,p)$ and the hamiltonian $H^\delta(x,p)$ is defined through an associated cell problem. In the case of infinite Drichlet boundary condition the convergence of H^δ is established up to subsequences. Once established the existence of a unique lipschitz continuous function V, the convergence results for u^ϵ is stated in the following theorems.

Theorem 1.3.1. *(Neumann B.C.) Let u^ϵ be the solution to (1.3.1) and (1.3.7). Then if $\delta = \epsilon^\alpha$ with $\alpha \in (0, \frac{1}{2})$, $A = A(x)$ and B a constant matrix then*

$$u^\epsilon \longrightarrow \begin{cases} 1 & \textit{locally uniformly in} \quad Int\{V=0\} \times \overline{\Omega} \\ 0 & \textit{locally uniformly in} \quad \{V>0\} \times \overline{\Omega} \end{cases}$$

and if $\delta = \epsilon^\alpha$ with $\alpha \in [\frac{1}{2}, 1)$, $A = A(x)$ and $B = B(x,y)$ then

$$u^\epsilon \longrightarrow \begin{cases} 1 & \textit{locally uniformly in} \quad Int\{V=0\} \times \overline{\Omega} \\ 0 & \textit{locally uniformly in} \quad \{V>0\} \times \overline{\Omega}. \end{cases}$$

In the case of Drichlet boundary conditions, the convergence holds up to subsequences

Theorem 1.3.2. *(Drichlet B.C) Let u^ϵ be the solution to (1.3.1) with (1.3.6). Then there is $\delta_j = \epsilon_j^\alpha$ where $\alpha \in (0, \frac{1}{2})$, $A = A(x)$ and B a constant matrix then*

$$u^{\epsilon_j} \longrightarrow \begin{cases} 1 & \textit{locally uniformly in} \quad Int\{V=0\} \times \Omega \\ 0 & \textit{locally uniformly in} \quad Int\{V>0\} \times \Omega \end{cases}$$

and if $\delta = \epsilon^\alpha$ with $\alpha \in [\frac{1}{2}, 1)$, $A = A(x)$ and $B = B(x,y)$ then

$$u^{\epsilon_j} \longrightarrow \begin{cases} 1 & \textit{locally uniformly in} \quad Int\{V=0\} \times \Omega \\ 0 & \textit{locally uniformly in} \quad Int\{V>0\} \times \Omega. \end{cases}$$

Chapter 2

Neumann boundary conditions

2.1 Asymptotics of the Neumann B.C.

The first case to be considered is the asymptotic behavior of the solutions u^ϵ of

$$\begin{aligned} u_t - \epsilon Tr[AD_x^2 u] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 u] + \frac{1}{\epsilon} f(x,y,u) = 0 & \quad \text{in} \quad Q_T \\ \langle Du, n \rangle = 0 & \quad \text{in} \quad \partial Q_T \\ u = u_0(x,y) & \quad \text{in} \quad \mathbb{R}^n \times \Omega. \end{aligned}$$

Under the logarithmic change of variable (1.3.4), the functions v^ϵ satisfy the equation (1.3.5). Given the assumptions made (1.3.2) and (1.3.3) on f, it is sufficient to consider the particular case $\bar{c}(x,y,u) = c(x,y) - bu$ for $b > 0$ and $c(x,y)$ given by (1.3.3) .

For $\epsilon > 0$, let L be the operator defined by

$$\begin{aligned} L[v] &= v_t - \epsilon Tr[AD_x^2 v] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 v] + Tr[AD_x v \otimes D_x v] \\ &\quad + \frac{\delta^2}{\epsilon^2} Tr[BD_y v \otimes D_y v] + c(x,y) - be^{-\frac{v}{\epsilon}} \end{aligned} \tag{2.1.1}$$

and v^ϵ be the solutions of

$$\begin{aligned} L[v] &= 0 && \text{in} \quad Q_T \\ \langle D_y v, n \rangle &= 0 && \text{in} \quad \partial Q_T \\ v(x,y,0) &= \begin{cases} -\epsilon \log u_0(x,y) & (x,y) \in \mathrm{Int}\{\ \mathrm{supp}\ u_0\} \\ \infty & \text{otherwise.} \end{cases} \end{aligned} \tag{2.1.2}$$

For $T > 0$ define

$$\overline{Q}_T = (\mathbb{R}^n \times \overline{\Omega} \times (0,T)) \cup (\{\mathrm{supp}\ u_0\} \times \{0\}).$$

The first step is to establish that $v^\epsilon(x,y,t)$ converges to $V(x,t)$ locally uniformly in $\overline{Q}_T$. Holder estimates for the family $\{v^\epsilon\}_{\epsilon>0}$ uniform in ϵ are obtained. In the case of HJBI equation analyzed by Bardi et al [11], this type of estimates are not necessary since a comparison principle suffices. In our case, these type of estimates are essential for our analysis. Without the uniform in ϵ Holder estimates for v^ϵ, comparison principles between viscosity solutions can not be guaranteed. The Holder regularity of v^ϵ is established in the next theorem.

Theorem 2.1.1. *For any $K \subset\subset \overline{Q}_T$, there exist C_K and $0 < \alpha_1 \leq 1$ such that*

$$\|v_t^\epsilon\|_{C^{\alpha_1}(K)} + \|v^\epsilon\|_{\infty,K} + \|D_x v^\epsilon\|_{\infty,K} + \frac{\delta}{\epsilon}\|D_y v^\epsilon\|_{\infty,K} \leq C_K$$

uniformly in ϵ.

The proof of this theorem is divided in the following two lemmas.

Lemma 2.1.1. For any $K \subset\subset \overline{Q}_T$, there exist a constant C_K such that

$$\|v^\epsilon\|_{\infty,K} \leq C_K$$

uniformly in ϵ.

Proof. Without loss of generality, assume $(0,0) \in \text{supp } u_0$ and take $r > 0$ such that $B_r(0,0) \subset\subset \{\text{supp } u_0\}$. On $B_r(0,0)$, define

$$z_1(x,y,t) = \frac{1}{r^2 - |x|^2} + \frac{\lambda}{r^2 - |y|^2} + \alpha t + \beta,$$

where the value of the parameters λ, α, $\beta > 0$ of z_1 will be chosen later. Then plugging z_1 into L

$$\begin{aligned}
L[z_1] &= \alpha - \epsilon\left[\frac{2Tr[A]}{(r^2-|x|^2)^2} + \frac{8Tr[Ax\otimes x]}{(r^2-|x|^2)^3}\right] + \frac{4Tr[Ax\otimes x]}{(r^2-|x|^2)^4} \\
&\quad -\frac{\delta^2\lambda}{\epsilon}\left[\frac{2Tr[B]}{(r^2-|y|^2)^2} + \frac{8Tr[By\otimes y]}{(r^2-|y|^2)^3}\right] + \frac{\delta^2\lambda^2}{\epsilon^2}\left[\frac{4Tr[By\otimes y]}{(r^2-|y|^2)^4}\right] \\
&\quad +c(x,y) - be^{-\frac{1}{\epsilon}z_1} \\
&\geq \frac{\alpha}{2} - \epsilon C\left[\frac{1}{(r^2-|x|^2)^2} + \frac{|x|^2}{(r^2-|x|^2)^3}\right] + \frac{4\mu|x|^2}{(r^2-|x|^2)^4} \\
&\quad +\frac{\delta^2\lambda^2}{\epsilon^2}\left[-\frac{\epsilon C}{\lambda}\left(\frac{1}{(r^2-|y|^2)^2} + \frac{|y|^2}{(r^2-|y|^2)^3}\right) + \frac{4\mu|y|^2}{(r^2-|y|^2)^4}\right] \geq 0.
\end{aligned}$$

for $\lambda = \frac{\epsilon}{\delta}$ and $\alpha = \alpha_r$ sufficiently large independent of ϵ. Choose

$$\beta = -\epsilon \log\left[\inf_{B_r(0,0)} u_0\right]$$

then, the maximum principle implies $z_1 \geq v^\epsilon$ on $B_r(0,0) \times (0,T)$ and then

$$|v^\epsilon| \leq C \qquad \text{on} \qquad B_{\frac{r}{2}}(0,0) \times (0,T).$$

On $B_{\frac{r}{2}}(0,0)^c \times (0,T)$, define

$$z_2 = \frac{\rho}{t}\left[|x|^2 + \frac{\epsilon}{\delta}e^{|y|^2}\right] + \alpha t + \beta.$$

The convexity of Ω leads that for $(x,y,t) \in \mathbb{R}^n \times \partial\Omega \times (0,T)$

$$\langle Dz_2, n\rangle \geq \frac{\epsilon\rho}{\delta t}e^{|y|^2}\langle y, n\rangle \geq 0.$$

Plugging z_2 into equation (2.1.2) yields

$$L[z_2] \geq I + II + III$$

where

$$\begin{aligned} I &= \alpha + c(x,y) - be^{-\frac{z}{\epsilon}} \\ II &= \frac{\rho}{t^2}\left[-2\epsilon t Tr[A] + 4\rho Tr[Ax \otimes x] - |x|^2\right] \\ III &= \frac{\rho}{t^2}e^{|y|^2}\left[-2\delta t Tr[B] + 4\rho\tau e^{|y|^2} Tr[By \otimes y] - \frac{\epsilon}{\delta}\right]. \end{aligned}$$

For α sufficiently large, we obtain $I \geq 0$. In $\{|x| > r\} \times \overline{\Omega} \times (0,T)$, the positivity of the matrix B yields

$$\begin{aligned} II &\geq \frac{\rho}{t^2}[-2\epsilon T\Lambda + |x|^2(4\mu\rho - 1)] \geq \frac{\rho}{t^2}[-2\epsilon T\Lambda + 2\rho r^2\mu] \geq 0 \\ III &\geq -2\frac{\rho}{t}e^{|\Omega|^2}\delta\Lambda \end{aligned}$$

for ρ sufficiently large and ϵ sufficiently small. Then on $\{|x| > r\} \times \overline{\Omega} \times (0,T)$

$$I + II + III \geq \frac{\rho}{t^2}[\rho r^2\mu - 2Te^{|\Omega|^2}\delta\Lambda] \geq 0$$

for a large value of ρ. In $\{|x| \leq r\} \times \{|y| > r\} \times (0,T)$

$$\begin{aligned} III &\geq \frac{\rho}{t^2}\left[-2\delta T\Lambda e^{|\Omega|^2} + 4\rho e^{2r^2}r^2 - \frac{\epsilon}{\delta}\right] \geq \frac{\rho^2}{t^2} \\ II &\geq -2\epsilon T\Lambda\frac{\rho}{t^2}, \end{aligned}$$

for sufficiently large ρ. On $\{|x| \le r\} \times \{|y| > r\} \times (0,T)$

$$I + II + III \ge \frac{\rho}{t^2}[\rho - 2\epsilon T\Lambda] \ge 0$$

for ϵ sufficiently small. Therefore z_2 is a supersolution in $(B_{\frac{r}{4}}(y_0, x_0))^c \times (0,T)$. Thus the maximum principle implies that

$$v^\epsilon \le \max[z_1, z_2] \qquad \text{in} \qquad \mathbb{R}^n \times \overline{\Omega} \times (0,T),$$

and the local bound for $\|v^\epsilon\|_\infty$ in $\mathbb{R}^n \times \overline{\Omega} \times (0,T)$ uniform in $\epsilon > 0$ follows. The proof of the bound up to $t = 0$ follows directly from the arguments in [15]. $\square$

Once we have shown that $\{v^\epsilon\}_\epsilon$ is uniformly bounded in compact subsets of $\overline{Q}_T$, local gradient bounds can be obtained through the classical Bernstein method.

Lemma 2.1.2. Let v^ϵ be the solution to (1.3.5)-(1.3.7) . Let K be a compact subset of Q_T, then there exists $C_K > 0$ and $0 < \alpha' < 1$ such that

$$\|v_t\|_{C^{\alpha'}(K)} + \|D_x v^\epsilon\|_{\infty,K} + \frac{\delta}{\epsilon}\|D_y v^\epsilon\|_{\infty,K} \le C_K$$

uniformly in ϵ.

Proof. Let $B_r(x_0, t_0) \subset \mathbb{R}^n \times (0,T)$, and $\eta(x,t)$ be a smooth function satisfying

- supp $\eta = B_r(x_0, t_0)$.
- $0 \le \eta \le 1$ in $B_r(x_0, t_0)$.
- $\eta = 1$ in $B_{\frac{r}{2}}(x_0, t_0)$.

Take $\zeta(x,y,t) = \eta(x,t)^4 e^{-\lambda v}$ where $\lambda > 0$ will be determined later. For $\tau > 0$, let

$$z = \zeta[|D_x v|^2 + \tau |D_y v|^2] = \zeta |Dv_\tau|^2.$$

At the boundary, the Neumann condition satisfied by v^ϵ leads to

$$\begin{aligned}\langle D_y z, n(y)\rangle &= -\lambda \eta^4 e^{-\lambda v} |Dv_\tau|^2 \langle D_y v, n(y)\rangle + 2\zeta[v_{x_k} v_{x_k y_i} n_i + \tau v_{y_k} v_{y_k y_i} n_i] \\ &= 2\zeta[v_{x_k} v_{x_k y_i} n_i + \tau v_{y_k} v_{y_k y_i} n_i]. \end{aligned} \tag{2.1.3}$$

Since for $y \in \partial\Omega$ there is a $\gamma \geq 0$ such that

$$0 = D_{x_i}[\langle D_y v, n\rangle] = v_{y_k x_i} n_k$$

$$\gamma n_i = D_{y_i}[\langle D_y v, n(y)\rangle] = v_{y_k y_i} n_k + v_{y_k} n_{k,y_i},$$

then (2.1.3), the Neumann boundary conditions, and the convexity of Ω yield

$$\langle D_y z, n(y)\rangle \leq -2\tau\zeta n_{k,y_i} v_{y_k} v_{y_i} \leq 0.$$

Hence z does not attain a maximum at $\mathbb{R}^n \times \partial\Omega \times (0,T)$. Plugging z into

$$L_1[z] = z_t - \epsilon Tr[AD_x^2 z] - \frac{\delta^2}{\epsilon}[BD_y^2 z]$$

we obtain that at an interior maximum $(x_\epsilon, y_\epsilon, t_\epsilon)$ of z

$$L_1[z]e^{\lambda v} = I + II + III + IV + V \geq 0 \tag{2.1.4}$$

where

$$\begin{aligned} I &= 2\eta^4[v_{x_k}(v_t - \epsilon A_{ij} v_{x_i x_j} - \frac{\delta^2}{\epsilon} B_{ij} v_{y_i y_j})_{x_k} + \tau v_{y_k}(v_t - \epsilon A_{ij} v_{x_i x_j} - \frac{\delta^2}{\epsilon} B_{ij} v_{y_i y_j})_{y_k}] \\ &\quad + 2\eta^4[v_{x_k}(\epsilon A_{ij,x_k} v_{x_i x_j} + \frac{\delta^2}{\epsilon} B_{ij,x_k} v_{y_i y_j}) + \tau v_{y_k}(\epsilon A_{ij,y_k} v_{x_i x_j} + \frac{\delta^2}{\epsilon} B_{ij,y_k} v_{y_i y_j})], \end{aligned}$$

and

$$\begin{aligned}
II &= -2\eta^4[\epsilon A_{ij}(v_{x_kx_j}v_{x_kx_i}+\tau v_{y_kx_j}v_{y_kx_i})+\frac{\delta^2}{\epsilon}B_{ij}(v_{x_ky_j}v_{x_ky_i}+\tau v_{y_ky_j}v_{y_ky_i})],\\
III &= -\lambda\eta^4[v_t-\epsilon A_{ij}v_{x_i,x_j}-\frac{\delta^2}{\epsilon}B_{ij}v_{y_iy_j}]|Dv_\tau|^2,\\
IV &= [(\eta^4)_t-\epsilon A_{ij}[(\eta^4)_{x_ix_j}-2\lambda(\eta^4)_{x_i}v_{x_j}]]|Dv_\tau|^2,\\
V &= -\lambda^2\eta^4[\epsilon A_{ij}v_{x_i}v_{x_j}+\frac{\delta^2}{\epsilon}B_{ij}v_{y_i}v_{y_j}]|Dv_\tau|^2\\
VI &= -\epsilon A_{ij}[(\eta^4)_{x_j}[v_{x_k}v_{x_kx_i}+\tau v_{y_k}v_{y_kx_i}]]\\
&\quad -2\lambda\eta^4(-\epsilon A_{ij}v_{x_j}[v_{x_k}v_{x_kx_i}+\tau v_{y_k}v_{y_kx_i}]-\frac{\delta^2}{\epsilon}B_{ij}v_{y_j}[v_{x_k}v_{x_ky_i}+\tau v_{y_k}v_{y_ky_i}])
\end{aligned}$$

Since v^ϵ satisfies (2.1.2) and A and B are symmetric matrices, then we have that $I=I_1+I_2$ where

$$\begin{aligned}
I_1 &= -4\eta^4[A_{ij}(v_{x_ix_k}v_{x_k}+\tau v_{x_iy_k}v_{y_k})v_{x_j}+\frac{\delta^2}{\epsilon^2}B_{ij}(v_{y_ix_k}v_{x_k}+\tau v_{y_iy_k}v_{y_k})v_{y_j}]\\
I_2 &\le -2\eta^4[A_{ij,x_k}v_{x_i}v_{x_j}v_{x_k}+\frac{\delta^2}{\epsilon^2}B_{ij,x_k}v_{y_i}v_{y_j}v_{x_k}\\
&\quad +\tau v_{y_k}(A_{ij,y_k}v_{x_i}v_{x_j}+\frac{\delta^2}{\epsilon^2}B_{ij,y_k}v_{y_i}v_{y_j})]\\
&\quad +2\eta^4[v_{x_k}(\epsilon A_{ij,x_k}v_{x_ix_j}+\frac{\delta^2}{\epsilon}B_{ij,x_k}v_{y_iy_j})\\
&\quad +\tau v_{y_k}(\epsilon A_{ij,y_k}v_{x_ix_j}+\frac{\delta^2}{\epsilon}B_{ij,y_k}v_{y_iy_j})]\\
&\quad -2\eta^4\langle Dc,Dv_\tau\rangle.
\end{aligned}$$

Denote by $A_x=\|D_xA\|$, $A_y=\|D_yA\|$, $B_x=\|D_xB\|$ and $B_y=\|D_yB\|$. Using Cauchy-Schwartz, we can find constants C and c sufficiently small independent of the parameter ϵ, such that the second order terms in the above expression can be estimated by

$$
\begin{aligned}
I_2 \leq C\eta^4[A_x|D_xv|^2 + \frac{\delta^2}{\epsilon^2}B_x|D_yv|^2|D_xv| + \tau(A_y|D_yv||D_xv|^2 + \frac{\delta^2}{\epsilon^2}B_y|D_yv|^3) \\
|D_xv|^2(\epsilon A_x + \frac{\delta^2}{\epsilon}B_x) + \tau|D_yv|^2(\epsilon A_y + \frac{\delta^2}{\epsilon}B_y) + |Dv_\tau|] \\
+ c(1+\tau)\eta^4(\epsilon|D_x^2v|^2 + \frac{\delta^2}{\epsilon}|D_y^2v|^2).
\end{aligned}
$$

Since at an interior maximum point of z we have

$$
\begin{aligned}
2\zeta^4[v_{x_m}v_{x_mx_n} + \tau v_{y_i}v_{y_jx_n}] &= -(\eta^4)_{x_n}|Dv_\tau|^2 + \lambda v_{x_n}|Dv_\tau|^2 \quad (2.1.5) \\
2\zeta^4[v_{x_m}v_{x_my_n} + \tau v_{y_m}v_{y_my_n}] &= -(\eta^4)_{y_n}|Dv_\tau|^2 + \lambda v_{y_n}|Dv_\tau|^2, \quad (2.1.6)
\end{aligned}
$$

then plugging (2.1.5) and (2.1.6) into I_1 we obtain

$$
I_1 \leq C\eta^3|Dv_\tau|^2|D_xv| - 2\lambda\eta^4(A_{ij}v_{x_i}v_{x_j} + \frac{\delta^2}{\epsilon^2}B_{ij}v_{y_i}v_{y_j})|Dv_\tau|^2.
$$

The strict positivity of A and B yield

$$
I \leq C\eta^3|Dv_\tau|^2|D_xv| - 2\lambda\mu\eta^4[|D_xv|^2 + \frac{\delta^2}{\epsilon^2}|D_yv|^2]|Dv_\tau|^2.
$$

Since v^ϵ satisfies (2.1.2) then

$$
\begin{aligned}
III \leq& \lambda[A_{ij}v_{x_i}v_{x_j} + \frac{\delta^2}{\epsilon^2}B_{ij}v_{y_i}v_{y_j} + c(x,y) - be^{-\frac{v}{\epsilon}}]|Dv_\tau|^2 \\
\leq& \lambda\eta^4[A_{ij}v_{x_i}v_{x_j} + \frac{\delta^2}{\epsilon^2}B_{ij}v_{y_i}v_{y_j}]|Dv_\tau|^2 + \lambda C\eta^4|Dv_\tau|^2.
\end{aligned}
$$

Now IV can be estimated by

$$
IV \leq C\eta^3|Dv_\tau|^2 + \epsilon C\lambda\eta^3|D_xv||Dv_\tau|^2
$$

and

$$
V \leq -\lambda^2\eta^4\mu[\epsilon|D_xv|^2 + \frac{\delta^2}{\epsilon^2}|D_yv|^2]|Dv_\tau|^2
$$

Using Cauchy-Schwartz inequality again at VI

$$\begin{aligned}VI \leq &\epsilon C\eta^3|Dv_\tau|^2 + \epsilon c\eta^4[|D_x^2 v|^2 + \tau|D_{xy}^2 v|^2] + \lambda C\eta^4|Dv_\tau|^2[\epsilon|D_x v|^2 + \frac{\delta^2}{\epsilon}|D_y v|^2] \\ &+ \lambda c\eta^4[\epsilon(|D_x^2 v|^2 + \tau|D_{xy}^2 v|^2) + \frac{\delta^2}{\epsilon}(|D_{xy}^2 v|^2 + \tau|D_y^2 v|^2)].\end{aligned}$$

Choosing c sufficiently small, the terms with second derivatives in I_2 and VI are cancelled by II. The previous estimates at $(x_\epsilon, y_\epsilon, t_\epsilon)$ and (2.1.4) yield, for ϵ sufficiently small,

$$\begin{aligned}\lambda\mu\eta^4[|D_x v|^2 + \frac{\delta^2}{\epsilon^2}|D_y v|^2]|Dv_\tau|^2 \leq\, &C\eta^4[A_x|D_x v|^2 + \frac{\delta^2}{\epsilon^2}B_x|D_y v|^2|D_x v| \\ &+ \tau(A_y|D_y v||D_x v|^2 + \frac{\delta^2}{\epsilon^2}B_y|D_y v|^3) \\ &+ |D_x v|^2(\epsilon A_x + \frac{\delta^2}{\epsilon}B_x) + \tau|D_y v|^2(\epsilon A_y + \frac{\delta^2}{\epsilon}B_y)].\end{aligned}$$

For $\lambda > 0$ sufficiently large and $\tau = \frac{\delta^2}{\epsilon^2}$, define $X = |D_x v|$ and $Y = \frac{\delta}{\epsilon}|D_y v|$ evaluated at $(x_\epsilon, y_\epsilon, t_\epsilon)$. Then

$$\lambda\mu\eta^4(X^2 + Y^2)^2 \leq B_1 + B_2,$$

where

$$B_1 = B_x Y^2 X + \frac{\delta}{\epsilon}A_y X^2 Y + \frac{\delta^2}{\epsilon}B_x X^2 + \frac{\delta^2}{\epsilon}B_y Y^2$$

and

$$B_2 = A_x X^2 + \frac{\epsilon}{\delta}B_y Y^3 + \epsilon A_x X^2 + \epsilon A_y Y^2.$$

Then the gradient bound

$$X + Y = |D_x v| + \frac{\delta}{\epsilon}|D_x v| \leq C \tag{2.1.7}$$

holds for $A = A(x)$ and B a constant positive matrix. If we restrict the size of δ with respect to ϵ, the gradient bounds can be obtained under more general coefficients. If $\delta = \epsilon^\alpha$ with $\alpha \in [\frac{1}{2}, 1)$, i.e $\frac{\delta^2}{\epsilon}$ remains bounded as $\epsilon \to 0$ and consider $A = A(x)$ and $B = B(x, y)$, then (2.1.7) holds at the maximum of z. By Lemma 2.1.1, v^ϵ is locally bounded, hence

$$|D_x v| + \frac{\delta}{\epsilon}|D_x v| \leq C \quad \text{in} \quad K = B_{\frac{r}{2}}(x_0, t_0) \times \overline{\Omega}. \tag{2.1.8}$$

The interior Holder estimate in t can be established as in [6] and [16] using (2.1.8). $\square$

Theorem 2.1.2 establishes the existence of a subsequence $v^{\epsilon_j} \to V$ locally uniformly in Q_T. The function $V = V(x, t)$ is Lipschitz in x and Holder continuous with exponent α_1 in t. In the next section, the function V is characterized as the unique solution of a variational inequality.

2.1.1 Cell problems

In order to formulate the equation satisfied by V, it is necessary to define the hamiltonian in the variational inequality. To define $H(x, p)$ in (2.1.21), the following cell problems should be considered.

For $x_0, p_0 \in \mathbb{R}^n$ and $\delta > 0$ define

$$F^\delta(D^2 w, Dw, y) = -\delta Tr[BD^2 w] + Tr[BDw \otimes Dw] + h(y)$$

where the function h is defined by

$$h(y) = Tr[A(x_0)p_0 \otimes p_0] + c(x_0, y) \qquad \text{in} \quad \overline{\Omega}.$$

For $\delta, r > 0$ and $\phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ smooth, define

$$\underline{F}_r^\delta(Y,q,y) = \min_{B_r(x_0,t_0)} \{-\delta Tr[BY] + Tr[Bq \otimes q] + Tr[AD\phi \otimes D\phi] + c(x,y)\} \tag{2.1.9}$$

and

$$\overline{F}_r^\delta(Y,q,y) = \max_{B_r(x_0,t_0)} \{-\delta Tr[BY] + Tr[Bq \otimes q] + Tr[AD\phi \otimes D\phi] + c(x,y)\}. \tag{2.1.10}$$

Consider χ^δ as the solution of

$$F^\delta(D^2\chi, D\chi, y) = H^\delta(x^0, p^0) \quad \text{in} \quad \Omega \tag{2.1.11}$$
$$\langle D\chi, n\rangle = 0 \quad \text{on} \quad \partial\Omega,$$

$\overline{\chi}_r^\delta$ to be the solution to

$$\overline{F}_r^\delta(D^2\chi, D\chi, y) = H_r^\delta(x_0, p_0) \quad \text{in} \quad \Omega \tag{2.1.12}$$
$$\langle D\chi, n\rangle = 0 \quad \text{on} \quad \partial\Omega,$$

and $\underline{\chi}_r^\delta$ be the solution to

$$\underline{F}_r^\delta(D^2\chi, D\chi, y) = H_r^\delta(x_0, p_0) \quad \text{in} \quad \Omega \tag{2.1.13}$$
$$\langle D\chi, n\rangle = 0 \quad \text{on} \quad \partial\Omega.$$

To show the existence of solutions to the cell problems (2.1.11)- (2.1.13), consider the following approximate cell problems.

For problem (2.1.11) and $\lambda > 0$, let $\chi^{\lambda,\delta}$ solve

$$\lambda\chi - \delta Tr[BD_y^2\chi] + Tr[BD\chi \otimes D\chi] + h(y) = 0 \quad \text{in} \quad \Omega \tag{2.1.14}$$
$$\langle D\chi, n\rangle = 0 \quad \text{on} \quad \partial\Omega.$$

For problem (2.1.12), let $\overline{\chi}_r^{\lambda,\delta}$ be the solution of

$$\begin{aligned} \lambda\chi + \overline{F}_r^{\delta}(D^2\chi, D\chi, y) &= 0 \qquad \text{in} \quad \Omega \\ \langle D\chi, n\rangle &= 0 \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{2.1.15}$$

and for problem (2.1.13), let $\underline{\chi}_r^{\lambda,\delta}$ be the solution of

$$\begin{aligned} \lambda\chi + \underline{F}_r^{\delta}(D^2\chi, D\chi, y) &= 0 \qquad \text{in} \quad \Omega \\ \langle D\chi, n\rangle &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.1.16}$$

Before establishing the existence of solutions for the above cell problems, we need gradient bound results used to obtain the properties needed for the cell problems.

Theorem 2.1.2. *There exist a $C > 0$ independent of $\lambda > 0$ and $\delta > 0$ such that if $\chi^{\delta,\lambda}$ satisfies*

$$\begin{aligned} \lambda\chi + F^{\delta}(D^2\chi, D\chi, y) = 0 \qquad &\textit{in} \quad \Omega \\ \langle D\chi, n\rangle = 0 \qquad &\textit{on} \quad \partial\Omega \end{aligned}$$

then

$$\|\chi^{\delta,\lambda}\|_{W^{1,\infty}(\overline{\Omega})} \leq C.$$

Proof. Under the assumption that for all λ, the term $\lambda\chi$ is bounded independently of λ in $\overline{\Omega}$. The proof follows from similar arguments as in [10].

The properties of the solutions to the cell problems introduced above are stated in the following lemma

Lemma 2.1.3. For $\delta > 0$, $r > 0$

a) There exist a unique $\chi^{\lambda,\delta} \in C^{2,\alpha}(\bar{\Omega})$ satisfying (2.1.14).

b) There exist a locally Lipschitz continuous functions $\overline{\chi}_r^{\delta,\lambda}$ and $\underline{\chi}_r^{\delta,\lambda}$ satisfying (2.1.15) and (2.1.16) in the viscosity sense.

c) For a point $y_0 \in \Omega$, $\chi^{\lambda,\delta} - \chi^{\lambda,\delta}(y_0)$ is bounded in $W^{1,\infty}(\overline{\Omega})$ uniformly in λ and δ. For $y_0 \in \Omega$, $x_0, p_0 \in \mathbb{R}^n$ and fixed δ, $\chi^{\lambda,\delta}(y) - \chi^{\lambda,\delta}(y_0) \to \chi^{\delta}(y)$ uniformly in $\overline{\Omega}$, and $-\lambda\chi^{\lambda,\delta} \to H^{\delta}(x^0, p^0)$ uniformly in $\overline{\Omega}$ as $\lambda \to 0^+$.

d) For $r > 0$ and $y_0 \in \Omega$, $\overline{\chi}_r^{\lambda,\delta} - \overline{\chi}_r^{\lambda,\delta}(y_0)$ is bounded in $W^{1,\infty}(\overline{\Omega})$ uniformly in λ and δ. For $y_0 \in \Omega$, $x_0, p_0 \in \mathbb{R}^n$ and fixed δ, $\overline{\chi}_r^{\lambda,\delta}(y) - \overline{\chi}_r^{\lambda,\delta}(y_0) \to \overline{\chi}_r^{\delta}(y)$ uniformly in $\overline{\Omega}$, and $-\lambda\overline{\chi}_r^{\lambda,\delta} \to \overline{H}_r^{\delta}(x_0, p_0)$ uniformly in $\overline{\Omega}$ as $\lambda \to 0^+$. Similar properties can be established for $\underline{\chi}_r^{\lambda,\delta}$, $\underline{\chi}^{\lambda,\delta}$ and $\overline{H}_r^{\delta}(x_0, p_0)$.

e) $H_r^{\delta}(x_0, p_0) \to H^{\delta}(x_0, p_0)$ as $r \to 0^+$. There exist a constant $H(x_0, p_0)$ such that $H^{\delta}(x_0, p_0) \to H(x_0, p_0)$ as $\delta \to 0^+$.

f) The function $H(x, p)$ satisfies the assumptions (H1-H3) in section 1.2.2.

Proof. Fix $\delta > 0$. The existence of a unique $\chi^{\lambda,\delta} \in C^{2,\alpha}(\overline{\Omega})$ follows by [17]. The existence of $\overline{\chi}_r^{\lambda,\delta}$ follows from Perron's method, since

$$\overline{w} = 0 \qquad \text{and} \qquad \underline{w} = -\max_{\overline{\Omega}} \frac{h(y)}{\lambda} \tag{2.1.17}$$

are a supersolution and a subsolution of (2.1.14)- (2.1.16) respectively. To obtain a bound for $\chi^{\lambda,\delta}$ in $L^{\infty}(\overline{\Omega})$ uniformly on λ and δ, take $y_0 \in \Omega$, and let

$$z^{\lambda,\delta} = \frac{\chi - \chi(y_0)}{\|\chi - \chi(y_0)\|_{\infty,\bar{\Omega}}}.$$

Then for all δ, λ, $z^{\lambda,\delta}$ satisfies $\|z\|_\infty = 1$, $z(y_0) = 0$ and

$$\lambda(\tau z + z_\lambda) - \tau\delta Tr[BD^2 z] + \tau^2 Tr[BDz \otimes Dz] + h(y) = 0 \quad \text{in} \quad \Omega$$
$$\langle Dz, n\rangle = 0 \quad \text{in} \quad \partial\Omega$$

where $\tau = \|\chi^\lambda - \chi^\lambda(y_0)\|_{\infty,\bar{\Omega}}$ and $z_\lambda = \chi_\lambda(y_0)$. If we assume that $\tau_\lambda \to \infty$ as $\lambda \to 0^+$. By the uniform in λ and δ lipschitz bounds for $z^{\lambda,\delta}$ in $\overline{\Omega}$ given in the Theorem 2.1.2, then $z^{\lambda_i,\delta} \to Z^\delta$ uniformly in $\overline{\Omega}$, and Z^δ satisfies

$$Tr[BDZ \otimes DZ] = 0 \quad \text{in} \quad \Omega$$
$$\langle DZ, n\rangle = 0 \quad \text{in} \quad \partial\Omega.$$

Since $z^{\lambda,\delta}(y_0) = 0$ for all λ, then $Z^\delta(y_0) = 0$. This implies $Z^\delta = 0$ in $\bar{\Omega}$. This contradicts the fact that $\|z^{\lambda,\delta}\|_\infty = 1$ and $z^{\delta,\lambda} \to Z^\delta$. Hence the function $W^{\lambda,\delta} = \chi^{\lambda,\delta} - \chi^{\lambda,\delta}(y_0)$ is bounded in $\overline{\Omega}$ independent of λ and δ. Now $W^{\lambda,\delta}$ satisfies

$$\lambda(W + z_\lambda) - \delta Tr[BD^2 W] + Tr[BDW \otimes DW] + h(y) = 0 \quad \text{in} \quad \Omega$$
$$\langle DW, n\rangle = 0 \quad \text{on} \quad \partial\Omega.$$

The uniform in λ and δ gradient bound for $W^{\lambda,\delta}$ in $\bar{\Omega}$ follows from the lipschitz bounds in [15]. Since $\{\lambda\chi^{\delta,\lambda}(y_0)\}_{\lambda>0}$ is bounded, then there exist a sequence $\lambda_j \to 0$ for which $-\lambda\chi^{\lambda,\delta}(y_0) \to H^\delta$. Let C be the Lipschitz constant for $W^{\lambda,\delta}$, then

$$|\lambda\chi(y) - \lambda\chi(y_0)| = \lambda|W^\delta(y) - W^\delta(y_0)| \leq \lambda C|y - y_0| \to 0 \quad \text{in } \overline{\Omega}$$

as $\lambda \to 0$. Then $-\lambda\chi^{\lambda,\delta} \to H^\delta$ uniformly in $\overline{\Omega}$ as $\lambda_j \to 0$. The uniqueness of H^δ follows by similar arguments as the uniqueness proof for H shown below. Hence $-\lambda w^{\lambda,\delta} \to H^\delta$ uniformly in $\overline{\Omega}$. Similar arguments as in the case of equation (2.1.14) establish the existence of a unique H_r^δ. The stability property of viscosity solutions implies that $H_r^\delta \to H^\delta$ as $r \to 0^+$. Since $\underline{w} \le \chi^{\delta,\lambda} \le \overline{w}$ in $\overline{\Omega}$, where $\underline{w}$ and $\overline{w}$ are given by (2.1.17), then $0 \le \lambda\chi^{\delta,\lambda} \le \|h\|_{\infty,\overline{\Omega}}$. The boundedness of h in $\overline{\Omega}$ implies H^δ is bounded uniformly in δ. The gradient bounds in Theorem 2.1.2 imply that there exist a sequence $\delta_j \to 0$ such that χ^δ converge locally uniformly to χ in $\overline{\Omega}$ and $H^{\delta_j} \to H$. The function χ satisfies

$$Tr[BD\chi \otimes \chi] + h(y) = H \quad \text{in} \quad \Omega \tag{2.1.18}$$

$$\langle D\chi, n\rangle = 0 \quad \text{in} \quad \partial\Omega. \tag{2.1.19}$$

Let (χ_1, H_1) and (χ_2, H_2) be two solutions to (2.1.18) and (2.1.19). Assume that $H_2 < \beta < H_2$. Since χ_1 and χ_2 are bounded and Lipschitz in $\overline{\Omega}$, by adding a constant if needed, assume $\chi_1 > \chi_2$ in $\overline{\Omega}$. For $\epsilon > 0$ sufficiently small χ_1 satisfies

$$Tr[BD\chi_1 \otimes \chi_1] + h(y) + \epsilon\chi_1 \le \beta \quad \text{in } \Omega$$
$$\langle D\chi_1, n\rangle \le 0 \quad \text{in } \partial\Omega,$$

and χ_2 satisfies

$$Tr[BD\chi_2 \otimes \chi_2] + h(y) + \epsilon\chi_2 \ge \beta \quad \text{in } \Omega$$
$$\langle D\chi_1, n\rangle \ge 0 \quad \text{in } \partial\Omega.$$

Hence χ_1 and χ_2 are Lipschitz continuous subsolution and supersolution to

$$Tr[BD\chi \otimes \chi] + h(y) + \epsilon\chi = \beta \quad \text{in} \quad K$$
$$\langle D\chi, n\rangle = 0 \quad \text{in} \quad \partial K.$$

Then a comparison principle for Lipschitz continuous viscosity solutions of this equation establish that $\chi_1 \leq \chi_2$ in $\overline{\Omega}$, a contradiction. Hence there is a unique value H, such that $\lim_{\delta\to 0^+} H^\delta = H$.

To show f), define for fixed $y \in \Omega$, the function

$$G(x,p) = Tr[Ap \otimes p] + c(x,y).$$

Then it is easy to check that G satisfies H1-H3. Let $h(y) = G(x,p)$ in (2.1.14) then by similar arguments as in [5], these properties hold for $H(x,p)$ $\square$

In the particular case that $c = c(x)$, for $\delta > 0$, consider the function $\varphi = e^{-\frac{1}{\delta}w^\delta}$, where w^δ is the solution to the cell problem (2.1.11). Then φ satisfies

$$\begin{aligned} -\delta^2 Tr[BD^2\varphi] &= \lambda\varphi \quad \text{in} \quad \Omega \\ \langle D\varphi, n\rangle &= 0 \quad \text{in} \quad \partial\Omega. \end{aligned} \tag{2.1.20}$$

and $\varphi > 0$ in Ω. Hence in this case, the Hamiltonian H^δ is given by

$$H^\delta(x,p) = Tr[Ap \otimes p] + c(x) - \lambda^\delta(x).$$

where $\lambda^\delta(x)$ is the constant defined by the eigenvalue problem (2.1.20).

2.1.2 Asymptotical behavior of v^ϵ

Using the results in the last subsection, we can establish the equation satisfied by each one of the uniform limits of v^ϵ. The next theorem characterizes all the locally uniform limits of the collection $\{v^\epsilon\}_{\epsilon>0}$.

Theorem 2.1.3. *Let v^ϵ be the solution of (1.3.5), then if v^{ϵ_j} converges uniformly to V, V is a viscosity solution of*

$$\max[V_t + H(x, DV), V] = 0 \qquad \text{in } \mathbb{R}^n \times (0,T) \tag{2.1.21}$$

$$V(x,0) = \begin{cases} 0 & x \in Int\{x|\ \text{there exist } y \text{ s.t. } \quad u_0(x,y) > 0\} \\ \infty & x \in \overline{\{x|\ \text{there exist } y \text{ s.t. } \quad u_0(x,y) > 0\}}^c, \end{cases}$$

where $H(x,p) = \lim_{\delta\to 0} H^\delta(x,p)$, and $H^\delta(x,p)$ is defined through equation (2.1.11).

Proof. From Theorem 2.1.1, for all $(x_0, t_0) \in \mathrm{Int}\{\text{ supp } u_0\}$ we have

$$v^\epsilon(x,y,t) \le \min\left[\frac{1}{r^2 - |x-x_0|^2} + \frac{\epsilon}{\delta}\frac{1}{r^2 - |y-y_0|^2}, \frac{\rho}{t}\left[|x-x_0|^2 + \frac{\epsilon}{\delta}e^{|y-y_0|^2}\right]\right] + \alpha t + \beta$$

on $B_r(x_0) \times B_r(y_0) \times (0,T)$.

For $(x,y,t) \in (B_r(x_0) \times B_r(y_0))^c \times (0,T)$

$$-\beta\epsilon \le v^\epsilon(x,y,t) \le \frac{\rho}{t}\left[|x-x_0|^2 + \frac{\epsilon}{\delta}e^{|y-y_0|^2}\right] + \alpha t + \beta$$

then

$$V(x,0) = \lim_{t\to 0^+}\lim_{\epsilon\to 0} v^\epsilon(x,y,t) = 0$$

in $\{x \in \mathbb{R}^n \mid \exists y \in \Omega \text{ such that}(x,y) \in \text{Int } \{ \text{ supp } u_0\}\}$. Let $R > 0$ and $x_0 \in \mathbb{R}^n$ be such that

$$\text{supp } u_0 \subset B_R(x_0) \times \Omega.$$

Assume that $x_0 = 0$. Define for $\rho > 0$

$$z(x,y,t) = \frac{\rho}{t}[|x|^2 - R^2] - 2\epsilon\beta.$$

Then

$$L[z] \leq \frac{\rho}{t^2}[(4\rho - 1)|x|^2 + R^2] + c(x,y) \leq 0$$

on $\{|x| > R_1 > R\} \times \overline{\Omega} \times (0,t)$ and for sufficiently small $t > 0$ and $\rho = \rho(R_1)$. Hence $\underline{z}(x,y,t) = \max[z(x,y,t), -\epsilon\beta]$ is a subsolution for the operator L. This establishes

$$V(x,0) = \lim_{t \to 0^+} \lim_{\epsilon \to 0} v^\epsilon(x,y,t) = \infty$$

on $\{|x| > R\}$.

To establish the subsolution condition, let $V - \phi$ have a strict local maximum at $(x_0,t_0) \in \{V > 0\}$, with $t_0 > 0$ and $(V - \phi)(x_0,t_0) = 0$. Suppose that at (x_0,t_0) we have

$$\phi_t + H(x_0, D\phi) \geq \beta > 0.$$

Pick $\delta > 0$ sufficiently small so that

$$|H^\delta(x_0,p_0) - H(x_0,p_0)| < \frac{\beta}{10}$$

where $p_0 = D\phi(x_0, t_0)$. Take $r > 0$ sufficiently small such that

$$
\begin{aligned}
|\underline{H}_r^\delta(x_0, p_0) - H^\delta(x_0, p_0)| &< \frac{\beta}{10} \\
\|\phi_t - \phi_t(x_0, t_0)\|_{\infty, B_r(x_0, t_0)} &< \frac{\beta}{10} \\
\|D^2\phi(x, t) - D^2\phi(x_0, t_0)\|_{\infty, B_r(x_0, t_0)} &< \frac{\beta}{10}.
\end{aligned}
$$

Let $\phi^\epsilon = \phi(x, t) + \frac{\epsilon}{\delta}\chi(y)$, where $\chi = \underline{\chi}_r^\delta$. We claim that ϕ^ϵ is a supersolution of (2.1.2) in $B_r(x_0, t_0) \times \overline{\Omega}$. Let $(\bar{x}, \bar{y}, \bar{t})$ be a local minimum of $\phi^\epsilon - \psi$ in $B_r(x_0, t_0) \times \overline{\Omega}$. Then $\bar{y}$ is a local minimum of $\chi(y) - \bar{\psi}(y)$ in $\bar{\Omega}$, where $\bar{\psi}(y) = \frac{\delta}{\epsilon}(\psi(\bar{x}, y, \bar{t}) - \phi(\bar{x}, \bar{t}))$.

If $\bar{y} \in \partial\Omega$, then $\langle D\bar{\psi}, n\rangle \geq 0$. This implies that at $(\bar{x}, \bar{y}, \bar{t})$ we have $\langle D\psi, n\rangle \geq 0$. If $\bar{y} \in \Omega$, then

$$\underline{F}_r^\delta\left(\frac{\delta}{\epsilon}D_y^2\psi, \frac{\delta}{\epsilon}D_y\psi, y\right) \geq \underline{H}_r^\delta(x_0, p_0).$$

Plugging ψ evaluated at $(\bar{x}, \bar{y}, \bar{t})$ into equation (2.1.2), the above inequality leads to

$$
\begin{aligned}
L[\psi] &\geq \phi_t - \epsilon Tr[AD^2\phi] - be^{-\frac{\phi^\epsilon}{\epsilon}} + F^\delta\left(\frac{\delta}{\epsilon}D_y^2\psi, \frac{\delta}{\epsilon}D_y\psi, \bar{y}\right) \\
&\geq \phi_t(x_0, t_0) + \underline{F}_r^\delta\left(\frac{\delta}{\epsilon}D_y^2\psi, \frac{\delta}{\epsilon}D_y\psi, \bar{y}\right) - \frac{3\beta}{5} \\
&\geq \phi_t(x_0, t_0) + \underline{H}_r^\delta(x_0, p_0) - \frac{3\beta}{5} \\
&\geq \phi_t(x_0, t_0) + H(x_0, p_0) - \beta \geq 0.
\end{aligned}
$$

For the second inequality, the fact that $(x_0, t_0) \in \{V > 0\}$ is used to estimate the exponential term. Since χ^δ is bounded in $\overline{\Omega}$ uniformly in δ, then there exist $\alpha > 0$ such that $v^\epsilon - \psi^\epsilon < -\alpha$ in $\partial B_r(x_0, t_0) \times \overline{\Omega}$. Since v^ϵ and ψ^ϵ are subsolution and supersolution of (2.1.2) , then $v^\epsilon - \psi^\epsilon < -\alpha$ in $B_r(x_0, t_0) \times \overline{\Omega}$ by a comparison principle for Lipschitz subsolutions and supersolutions of this equation. The last estimate yields

$$0 = (V - \phi)(x_0, t_0) = \lim_{\substack{\epsilon \to 0^+ \\ (x.t) \to (x_0, t_0)}} (v^\epsilon - \psi^\epsilon) < -\alpha,$$

a contradiction. Hence V satisfies

$$V_t + H(x, DV) \leq 0 \text{ in } \{V > 0\}$$

in the viscosity sense.

The proof for the supersolution case follows from similar arguments. In this case, ψ^ϵ is constructed using $\chi = \overline{\chi}_r^\delta$. Since the proof follows the similar arguments as the subsolution case, we omit its proof. $\square$

Since V is Lipschitz, the same considerations as in [15] establish that V is given by the formula (1.2.7).

2.1.3 Asymptotics of u^ϵ

The following theorem establishes the asymptotic behavior u^ϵ which satisfies

$$\begin{aligned} u_t - \epsilon Tr[AD_x^2 u] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 u] + \frac{1}{\epsilon} f(x, y, u) = 0 \qquad & \text{in} \quad Q_T \\ \langle Du, n \rangle = 0 \qquad & \text{in} \quad \partial Q_T \\ u = u_0(x, y) \qquad & \text{in} \quad \mathbb{R}^n \times \Omega \end{aligned}$$

Theorem 2.1.4. *If $\delta = \epsilon^\alpha$ with $\alpha \in (0, \frac{1}{2})$, $A = A(x)$ and B a constant matrix then*

$$u^\epsilon \longrightarrow \begin{cases} 1 & \textit{locally uniformly in} \quad \textit{Int}\,\{V = 0\} \times \overline{\Omega} \\ 0 & \textit{locally uniformly in} \quad \{V > 0\} \times \overline{\Omega}, \end{cases}$$

and if $\delta = \epsilon^\alpha$ with $\alpha \in [\frac{1}{2}, 1)$, $A = A(x)$ and $B = B(x, y)$ then

$$u^\epsilon \longrightarrow \begin{cases} 1 & \textit{locally uniformly in} \quad \textit{Int}\,\{V = 0\} \times \overline{\Omega} \\ 0 & \textit{locally uniformly in} \quad \{V > 0\} \times \overline{\Omega} \end{cases}$$

where V is the viscosity solution of

$$\max[V_t + H(x, DV), V] = 0 \qquad \textit{in } \mathbb{R}^n \times (0, T)$$

$$V(x, 0) = \begin{cases} 0 & x \in \textit{Int}\{x|\ \textit{there exist } y \textit{ s.t.} \quad u_0(x, y) > 0\} \\ \infty & x \in \overline{\{x|\ \textit{there exist } y \textit{ s.t.} \quad u_0(x, y) > 0\}}^c \end{cases}$$

and $H(x, p) = \lim_{\delta \to 0} H^\delta(x, p)$ where H^δ is defined by the cell problem (2.1.11).

Proof. Fix $\tau > 0$ and define $\bar{f} : \mathbb{R}^n \times \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\bar{f}(x, y, u) = (c(x, y) + \tau)u - bu^2$$

then for $\tau \geq 0$, $\bar{f}(x, y, u) \geq f(x, y, u)$ for $(x, y) \in \mathbb{R}^n \times \Omega$ and $0 \leq u \leq 1$. If $\overline{u}^\epsilon_\tau$ is the solution to

$$\begin{aligned} & u_t - \epsilon Tr[AD_x^2 u] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 u] + \frac{1}{\epsilon}\bar{f}(x, y, u) = 0 && \text{in} \quad Q_T \\ & \langle D_y u, n \rangle = 0 && \text{in } \partial Q_T. \\ & u(x, y, 0) = u_0(x, y) && \text{in} \quad \mathbb{R}^n \times \Omega \end{aligned}$$

by the maximum principle we have

$$\overline{u}^\epsilon_\tau \geq u^\epsilon \qquad \text{in} \quad \mathbb{R}^n \times \overline{\Omega} \times [0, T]. \tag{2.1.22}$$

Take $\overline{v}^\epsilon_\tau = -\epsilon \log \overline{u}^\epsilon_\tau$. By Theorem 2.1.3, there exist a function $\bar{v}_\tau$ such that $\overline{v}^\epsilon_\tau \to \overline{v}_\tau$ locally uniformly in $\mathbb{R}^n \times \overline{\Omega} \times (0,T)$ as $\epsilon \to 0^+$. On $\{\overline{v}_\tau > 0\} \times \overline{\Omega}$, $\overline{u}^\epsilon_\tau \to 0^+$ locally uniformly. Now as $\tau \to 0^+$, the stability property of viscosity solutions imply $\overline{v}_\tau \to V$ locally uniformly in $\mathbb{R}^n \times (0,T)$. By (2.1.22) and the fact that $u^\epsilon_\tau \to 0$ locally uniformly in $\{\bar{v}_\tau > 0\} \times \overline{\Omega}$ then $u^\epsilon \to 0$ locally uniformly in $\{\bar{v}_\tau > 0\} \times \overline{\Omega}$ for all $\tau > 0$. Hence

$$u^\epsilon \to 0 \qquad \text{on} \quad \{V > 0\} \times \overline{\Omega}.$$

To analyze the behavior of u^ϵ in compact subsets of Int $\{V = 0\} \times \overline{\Omega}$, define $\underline{f} : \mathbb{R}^n \times \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\underline{f}(x,y,u) = c(x,y)u - bu^2$$

where

$$b = \frac{1}{2}\sup\{|f_{uu}(x,y,u)||(x,y) \in \mathbb{R}^n \times \Omega, 0 \le u \le 1\}.$$

Then

$$f(x,y,u) \ge \underline{f}(x,y,u) \qquad \text{for} \quad (x,y) \in \mathbb{R}^n \times \Omega \quad \text{and} \quad 0 \le u \le 1.$$

Let $\underline{u}^\epsilon$ be the solution to

$$\begin{aligned} u_t - \epsilon Tr[AD^2_x u] - \frac{\delta^2}{\epsilon} Tr[BD^2_y u] + \frac{1}{\epsilon}\underline{f}(x,y,u) &= 0 && \text{in} \quad Q_T \\ \langle D_y u, n\rangle &= 0 && \text{in} \quad \partial Q_T \\ u(x,y,0) &= u_0(x,y) && \text{in} \quad \mathbb{R}^n \times \Omega. \end{aligned}$$

Using the same arguments from the Theorem 2.1.3 the convergence of $\underline{v}^\epsilon$ to V uniformly on compact subsets of Q_T is obtained. First, we show that on compacts sets $C \times \overline{\Omega}$ of Int $\{\underline{V} = 0\} \times \overline{\Omega}$, there exist $\alpha > 0$ such that

$$\liminf u^\epsilon \geq \alpha > 0 \text{ uniformly in } C \times \overline{\Omega}. \tag{2.1.23}$$

Take $(x_0, t_0) \in$ Int $\{V = 0\}$ and assume $C = B_r(x_0) \times (t_0, t_0 + r)$ for some $r > 0$ such that $C \subset\subset$ Int $\{V = 0\}$. Define

$$\phi^\epsilon(x, t) = \phi(x, t) + \epsilon[1 - d(y)]$$

where $\phi(x, t) = |x - x_0|^2 + (t - t_0)^2$. Let $d(y)$ be a nonnegative function that equals to the distance function to $\partial\Omega$ in a neighborhood of $\partial\Omega$, with $0 < d < 1$ in Ω. Then $v^\epsilon - \phi^\epsilon$ has a local maximum at $(x^\epsilon, y^\epsilon, t^\epsilon) \in$ Int $C \times \overline{\Omega}$. If $y^\epsilon \in \partial\Omega$, then

$$\langle D_y\phi^\epsilon, n\rangle \geq \epsilon > 0$$

for sufficiently small $\epsilon > 0$. Hence at $(x^\epsilon, y^\epsilon, t^\epsilon)$

$$\phi^\epsilon_t - \epsilon Tr[AD^2_x\phi^\epsilon] - \frac{\delta^2}{\epsilon}Tr[BD^2_y\phi^\epsilon] + Tr[AD_x\phi^\epsilon \otimes D_x\phi^\epsilon] + \frac{\delta^2}{\epsilon^2}Tr[BD_y\phi^\epsilon \otimes D_y\phi^\epsilon] \leq b\underline{u}^\epsilon - c. \tag{2.1.24}$$

By the construction of ϕ^ϵ, there exist $m > 0$ such that

$$\underline{u}^\epsilon(x^\epsilon, y^\epsilon, t^\epsilon) \geq \frac{1}{b}[\inf_{C\times\overline{\Omega}} c - o_\epsilon(1)] > \alpha_1.$$

Since $(\underline{v}^\epsilon - \phi^\epsilon)(x^\epsilon, y^\epsilon, t^\epsilon) \geq (\underline{v}^\epsilon - \phi^\epsilon)(x_0, y, t_0)$ then we obtain

$$\underline{v}^\epsilon(x^\epsilon, y^\epsilon, t^\epsilon) \geq \underline{v}^\epsilon(x_0, y, t_0) - \epsilon(1 - d(y)).$$

Since $\underline{v}^\epsilon = -\epsilon \ln \underline{u}^\epsilon$ then we get

$$\underline{u}^\epsilon(x_0, y, t_0) \geq \underline{u}^\epsilon(x^\epsilon, y^\epsilon, t^\epsilon)e^{-(1-d(y))} \geq m > 0. \quad \text{for} \quad y \in \overline{\Omega}.$$

The maximum principle implies

$$u^\epsilon(x_0, y, t_0) \geq \underline{u}^\epsilon(x_0, y, t_0) \geq m > 0 \quad \text{in} \quad \overline{\Omega}.$$

The value of m is independent of (x_0, t_0), then (2.1.23) holds. The hypothesis on f imply that for $\eta \in (\frac{m}{2}, 1)$, there is a $\beta = \beta_{m,\eta} > 0$ such that

$$f(x, y, u) \leq \beta(1 - \eta - u)$$

for $u \in [\frac{m}{2}, 1]$ and $(x, y) \in B_r(x_0) \times \overline{\Omega}$. Let $\phi(x)$ be a smooth function that satisfies:

- $0 < \phi < 1 \quad$ in $B_r(x_0)$.
- supp $\phi = B_r(x_0)$.
- $\phi(x) = 1$ for $x \in B_{\frac{r}{2}}(x_0)$.

Define $f^\epsilon(t) = 1 - \eta - e^{-\frac{\tau}{\epsilon}(t-t_0)}$, where $\eta' \in (\frac{1}{2}, 1)$, and

$$z^\epsilon(x, y, t) = f^\epsilon(t)\phi(x) - \epsilon\gamma(t - t_0).$$

Let w^ϵ satisfy

$$\begin{aligned} w_t - \epsilon Tr[AD_x^2 w] - \frac{\delta^2}{\epsilon}Tr[BD_y^2 w] + \frac{\beta}{\epsilon}[w - 1 + \eta] &= 0 \quad && \text{in} \quad C \times \Omega \quad (2.1.25) \\ \langle D_y w, n \rangle &= 0 && \text{in} \quad C \cap \partial\Omega \\ w &= 0 && \text{in} \quad \partial C \times \overline{\Omega}. \end{aligned}$$

We claim that z^ϵ is a subsolution to this equation in $C \times \overline{\Omega}$. On the boundary of Ω, we have that $\langle D_y z^\epsilon, n \rangle = 0$. By construction of z, $z^\epsilon \leq 0$ for any $(x, y, t) \in (\partial B_r(x_0) \times \overline{\Omega} \times (t_0, t_0 + r)) \cup (\overline{B_r(x_0)} \times \overline{\Omega} \times \{t_0\})$. On the interior of $C \times \Omega$, we have

$$\begin{aligned} z_t^\epsilon - \epsilon Tr[AD_x^2 z^\epsilon] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 z] + \frac{\beta}{\epsilon}[z^\epsilon - 1 + \eta] &= -\epsilon\gamma - \epsilon f^\epsilon Tr[AD_x^2 \phi] \\ &+ \frac{\beta}{\epsilon}[f^\epsilon \phi - \epsilon\gamma(t - t_0) - 1 + \eta] + \frac{\tau}{\epsilon}\phi e^{-\frac{\tau}{\epsilon}(t-t_0)} \\ &\leq -\epsilon\gamma + O(\epsilon) + \frac{\tau}{\epsilon}\phi[-f^\epsilon + 1 - \eta] + \frac{\beta}{\epsilon}[f^\epsilon \phi - \epsilon\gamma(t - t_0) - 1 + \eta]. \end{aligned}$$

For sufficiently large $\gamma > 0$ and for $0 < \tau < \beta$

$$\begin{aligned} z_t^\epsilon - \epsilon Tr[AD_x^2 z^\epsilon] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 z] + \frac{\beta}{\epsilon}[z^\epsilon - 1 + \eta] &\leq -\frac{\tau}{\epsilon}(1 - \eta)[1 - \phi] \\ &+ \frac{\beta - \tau}{\epsilon}[f\phi - 1 + \eta] \leq -\frac{\tau}{\epsilon}(1 - \eta)[1 - \phi] \leq 0. \end{aligned}$$

Hence z^ϵ is a subsolution to (2.1.25) in $C \times \overline{\Omega}$. The maximum principle implies

$$u^\epsilon \geq w^\epsilon \geq z^\epsilon \quad \text{in} \quad C \times \overline{\Omega}.$$

Hence

$$\liminf_{\epsilon \to 0} u^\epsilon \geq \liminf_{\epsilon \to 0} z^\epsilon \geq (1 - \eta) \quad \text{in} \quad B_{\frac{r}{2}}(x_0) \times \overline{\Omega} \times (t_0, t_0 + r).$$

Since η is arbitrary, then

$$\liminf_{\epsilon \to 0} u^\epsilon \geq 1$$

locally uniformly in $B_{\frac{r}{2}}(x_0) \times \overline{\Omega} \times (t_0, t_0 + r)$. The fact that $u^\epsilon \leq 1$ in $\mathbb{R}^n \times \overline{\Omega} \times [0, T]$ allows to conclude that $u^\epsilon \to 1$ locally uniformly in Int $\{V > 0\} \times \overline{\Omega}$. $\square$

Chapter 3

Dirichlet boundary conditions

3.1 Asymptotic under Dirichlet conditions

Let the function u^ϵ be the unique solution to the equation

$$u_t - \epsilon Tr[AD_x^2 u] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 u] + \frac{1}{\epsilon} f(x,y,u) = 0 \quad \text{in} \quad Q_T \tag{3.1.1}$$

$$u(x,y,t) = 0 \quad \text{in} \quad \partial Q_T \tag{3.1.2}$$

$$u(x,y,0) = u_0(x,y) \quad \text{in} \quad \mathbb{R}^n \times \overline{\Omega}.$$

Consider the function v^ϵ that satisfies

$$L[v] = 0 \quad \text{in} \quad Q_T \tag{3.1.3}$$

$$v^\epsilon = \infty \quad \text{in} \quad \partial Q_T$$

$$v(x,y,0) = \begin{cases} -\epsilon \log u_0(x,y) & (x,y) \in \text{Int}\{ \text{ supp } u_0\} \\ \infty & \text{otherwise,} \end{cases}$$

where the operator L is defined as in (2.1.1). The Dirichlet boundary conditions introduce a new difficulty not present in the Neumann boundary condition case. For $t > 0$, the function $v^\epsilon(x,y,t)$ is not bounded in compact sets containing $\partial\Omega$. Then, gradient bounds of the type of Theorem 2.1.1, do not hold up to the boundary of the cylinder. This fact limit our convergence result to hold only up to subsequences of v^ϵ. In the next theorem the local interior $W^{1,\infty}_{loc}$ bounds for v^ϵ uniform in ϵ are established.

Theorem 3.1.1. *For any $K \subset\subset Q_T$, there exist constants C_K and $0 < \alpha_1 < 1$ such that*

$$\|v_t\|_{C^{\alpha_1}(K)} + \|v^\epsilon\|_{\infty,K} + \|D_x v^\epsilon\|_{\infty,K} + \frac{\delta}{\epsilon}\|D_y v^\epsilon\|_{\infty,K} \leq C_K$$

uniformly in ϵ.

The proof of this theorem is presented in the following two lemmas.

Lemma 3.1.1. Let v^ϵ be the solution to (3.1.3). Let $K \subset\subset Q_T$, then there exists $C = C_K > 0$ such that $\|v^\epsilon\|_{\infty,K} \leq C$ uniformly in ϵ.

Proof. Suppose $(0,0) \in$ Int supp u_0, and choose $r > 0$ such that the ball $B_r(0,0) \subset\subset$ Int supp u_0. Take $z_1(x,y,t)$ to be the supersolution of (2.1.2) constructed in Lemma 2.1.1 in $B_r(0,0) \times [0,T]$. Then z_1 provides a uniform bound in ϵ for v^ϵ in $B_{\frac{r}{2}}(0,0)\times[0,T]$. To obtain bounds for v^ϵ outside $B_{\frac{r}{2}}(0,0)\times[0,T]$, let $\gamma > 0$ and define $\Omega_\gamma = \{y \in \Omega |\ \text{dist}(y, \partial\Omega) < \gamma\}$, where γ will be chosen later. Take $d(y)$ to be a smooth function such that $d(y) = dist(y, \partial\Omega)$ in Ω_γ. Take α and β as in Lemma 2.1.1. Then for M, $\rho > 0$ to be determined later, define $z(x,y,t)$ by

$$z = \frac{1}{t}\left[M|x|^2 + \frac{\epsilon}{\delta}\frac{e^{\tau|y|^2}}{d(y)}\right] + \alpha t + \beta. \tag{3.1.4}$$

Plug z into equation (3.1.3)

$$L[z] = I + II + III$$

where

$$
\begin{aligned}
I &= \alpha + c(x,y) - be^{\frac{z}{\epsilon}} > 0, \\
II &= -\epsilon Tr[AD_x^2 z] + Tr[AD_x z \otimes D_x z] - \frac{M}{t^2}|x|^2 \\
&= \frac{4M^2}{t^2} Tr[Ax \otimes x] - \frac{M}{t^2}|x|^2 - 2\epsilon \frac{M}{t} Tr[A] \\
&\geq \frac{M}{t^2}[(4\mu M - 1)|x|^2 - 2\epsilon T \Lambda n] \geq \frac{M^2}{t^2}[|x|^2 - O(\epsilon)],
\end{aligned}
$$

and

$$
\begin{aligned}
III &= -\frac{\delta^2}{\epsilon} Tr[BD_y^2 z] + \frac{\delta^2}{\epsilon^2} Tr[BD_y z \otimes D_y z] - \frac{\epsilon}{\delta} \frac{1}{t^2 d(y)} e^{\tau|y|^2} \\
&\geq \frac{e^{\tau|y|^2}}{t^2 d^4}[-\delta t[2dTr[BDd \otimes Dd] - 4\tau d^2 Tr[BDd \otimes y] - d^2 Tr[BD^2 d]] \\
&\qquad + e^{\tau|y|^2} \mu |2\tau d(y) y - Dd|^2].
\end{aligned}
$$

The convexity of Ω implies that for sufficiently small γ, $\langle y, -Dd \rangle > c > 0$ on $\mathbb{R}^n \times \Omega_\gamma$. This shows that

$$
|2\tau d(y) y - Dd|^2 > c_0 > 0 \qquad \text{on } \mathbb{R}^n \times \Omega_\gamma \times (0,T)
$$

for any value of $\tau > 0$. Then

$$
\begin{aligned}
III &\geq \frac{e^{\tau|y|^2}}{t^2 d^4}[O(\delta) + \mu e^{\tau|y|^2} c_0^2] \geq \frac{c_0^2 e^{2\tau r^2}}{t^2 d^4} \\
II &\geq -\epsilon \frac{C}{t^2}.
\end{aligned}
$$

Hence $I + II + III > 0$, for ϵ sufficiently small on $\mathbb{R}^n \times \Omega_\gamma \times (0,T)$. Choose

$$
\tau > \frac{\max_{\overline{\Omega}} |Dd|}{\gamma r}
$$

then it follows that

$$|2\tau d(y)y - Dd| > c_1 > 0$$

on $\mathbb{R}^n \times (\{d(y) \geq \gamma\} \cap \{|y| > r\}) \times (0,T)$ and we obtain $I + II + III > 0$. For $x \in \{|x| > r\}$, $y \in \{|y| < r\}$ and $t \in (0,T)$

$$II \geq \frac{M^2 r^2}{t^2} \qquad \text{and} \qquad III \geq -\delta \frac{e^{\tau|y|^2}}{t\gamma^4}$$

and for $M = M_{\tau,\gamma}$ sufficiently large, $I+II+III > 0$. Then z is a supersolution in $B_r(0,0)^c \times [0,T]$. By the maximum principle

$$v^\epsilon \leq \min[z_1, z] \qquad \text{on} \quad \mathbb{R}^n \times \Omega \times [0,T].$$

This establishes the local bound for v^ϵ uniformly in ϵ. □

The next lemma establishes the bounds on the space and time derivatives of v^ϵ.

Lemma 3.1.2. Let v^ϵ be the solution to (3.1.3) . Let K be a compact subset of Q_T, then there exists $C_K > 0$ such that

$$\|v_t\|_{C^{\alpha_1}(K)} + \|D_x v^\epsilon\|_{\infty,K} + \frac{\delta}{\epsilon}\|D_y v^\epsilon\|_{\infty,K} \leq C_K$$

uniformly in ϵ.

Proof. Let $B_r(x_0,t_0) \subset \mathbb{R}^n \times (0,T)$. Take η and τ as in Lemma 2.1.2 and $z(x,y,t) = \eta(x,t)^4 e^{-\lambda e^v}[|D_x v|^2 + \tau|D_y v|^2] = \eta(x,t)^4 e^{-\lambda e^v}|Dv_\tau|^2$. Since $|Dv^\epsilon| \to \infty$ as $y \to \partial\Omega$. The first step is to show that for $\epsilon > 0$, z is bounded above in $\mathbb{R}^n \times \overline{\Omega} \times (0,T)$. In order to achieve this, we need to obtain upper an lower boundary estimates for v^ϵ and $|Dv^\epsilon|$. From the estimates in the previous lemma, we have that

$$\limsup_{y\to\partial\Omega} \quad d(y)v^\epsilon(x,y,t) \leq C_K \qquad \text{for } K \subset\subset \mathbb{R}^n \times \overline{\Omega} \times (0,T)$$

where d is the distance function to the boundary of Ω . Define for $C > 0$

$$w(x,y,t) = -\epsilon^2 \log\left[\frac{2d(y)}{\epsilon}\right]$$

Then for all $\epsilon > 0$, $\lim_{y\to\partial\Omega} w = \infty$. Now plugging w into (3.1.3) we have that for ϵ sufficiently small

$$\begin{aligned} L[w] =& \frac{\epsilon\delta^2}{d^2}[\epsilon(dTr[BD^2d] - Tr[BDd \otimes Dd]) + \epsilon^2 Tr[BDd \otimes Dd] \\ &+ \frac{\epsilon}{\delta^2} d^2(c(x,y) - be^{-\frac{w}{\epsilon}})] \\ \leq& \frac{\epsilon^2\delta^2}{d^2}[-\mu + \epsilon\Lambda + \frac{\epsilon^2}{\delta^2}c(x,y)] \leq 0 \end{aligned}$$

on $\{d(y) < \epsilon\} \times \mathbb{R}^n \times (0,T)$. Since

$$w = -\epsilon^2 \log 2 < 0 \leq -\epsilon \log \|u_0\| \qquad \text{in} \quad \mathbb{R}^n \times \{d(y) = \epsilon\} \times (0,T)$$

then $\underline{w} = \max[w, -\epsilon \log \|u_0\|]$ is a subsolution to (3.1.3) in $\mathbb{R}^n \times \Omega \times [0,T]$. Maximum principle implies that

$$v^\epsilon \geq \underline{w} \qquad \text{in} \qquad \Omega \times \mathbb{R}^n \times [0,T] \tag{3.1.5}$$

Hence close enough to $\partial\Omega$, the above estimate leads to

$$z^\epsilon(x,y,t) \leq Ke^{-\lambda\left(\frac{2\epsilon}{d}\right)^{\epsilon^2}} d(y)^{-4}$$

for some $K > 0$ independent of ϵ. Then $z \to 0$ as $y \to \partial\Omega$. Since v is locally bounded in $\Omega \times \mathbb{R}^n \times (0, T)$, then for each ϵ, z is bounded above. Let $(x_\epsilon, y_\epsilon, t_\epsilon)$ be a maximum of z in supp $\eta \times \overline{\Omega}$. If $y_\epsilon \in \partial\Omega$ then $|Dv| = 0$ in supp $\eta \times \Omega$. Then we can assume that $y_\epsilon \in \Omega$. Plugging z into (3.1.3) we obtain

$$L[z]e^{\lambda e^v} = I + II + III + IV + V + VI$$

where

$$\begin{aligned}
I &= 2\eta^4[v_{x_k}(v_t - \epsilon A_{ij}v_{x_ix_j} - \frac{\delta^2}{\epsilon}B_{ij}v_{y_iy_j})_{x_k} \\
&\quad +\tau v_{y_k}(v_t - \epsilon A_{ij}v_{x_ix_j} - \frac{\delta^2}{\epsilon}B_{ij}v_{y_iy_j})_{y_k}] \\
&\quad +2\eta^4[\epsilon A_{ij,x_k}v_{x_ix_j}v_{x_k} + \tau v_{y_k}(\epsilon A_{ij,y_k}v_{x_ix_j} + \frac{\delta^2}{\epsilon}B_{ij,y_k}v_{y_iy_j})] \\
II &= -2\eta^4[\epsilon A_{ij}(v_{x_kx_j}v_{x_kx_i} + \tau v_{y_kx_j}v_{y_kx_i}) + \frac{\delta^2}{\epsilon}B_{ij}(v_{x_ky_j}v_{x_ky_i} + \tau v_{y_ky_j}v_{y_ky_i})] \\
III &= -\lambda e^v\eta^4[v_t - \epsilon A_{ij}v_{x_i,x_j} - \frac{\delta^2}{\epsilon}B_{ij}v_{y_iy_j}]|Dv_\tau|^2 \\
IV &= [(\eta^4)_t - \epsilon A_{ij}[(\eta^4)_{x_ix_j} - 2\lambda e^v(\eta^4)_{x_i}v_{x_j}]]|Dv_\tau|^2 \\
V &= -\lambda e^v(\lambda e^v - 1)\eta^4[\epsilon A_{ij}v_{x_i}v_{x_j} + \frac{\delta^2}{\epsilon}B_{ij}v_{y_i}v_{y_j}]|Dv_\tau|^2 \\
VI &= -\epsilon A_{ij}[(\eta^4)_{x_j}[v_{x_k}v_{x_kx_i} + \tau v_{y_k}v_{y_kx_i}]] \\
&\quad -\lambda e^v\eta^4(-\epsilon A_{ij}v_{x_j}[v_{x_k}v_{x_kx_i} + \tau v_{y_k}v_{y_kx_i}] - \frac{\delta^2}{\epsilon}B_{ij}v_{y_j}[v_{x_k}v_{x_ky_i} + \tau v_{y_k}v_{y_ky_i}]).
\end{aligned}$$

Similar computations as in Lemma 2.1.2 yield that if $A = A(x)$, B is a constant matrix and $\delta = \epsilon^\alpha$ with $\alpha \in (0, \frac{1}{2})$ or $A = A(x)$, $B = B(x, y)$ and $\alpha \in [\frac{1}{2}, 1)$, then there exist a $C > 0$ independent of ϵ such that

$$\eta(|D_xv| + \tau|D_yv|) \leq C \qquad \text{at } (x_\epsilon, y_\epsilon, t_\epsilon).$$

Since v is locally bounded in $\Omega \times \mathbb{R}^n \times (0,T)$, then for each $K \subset\subset \Omega$ there exist a $c_K > 0$ such that

$$e^{\lambda e^{-v^\epsilon}} \geq c_K \quad \text{in} \quad K \times B_r(x_0, t_0)$$

Then for all ϵ sufficiently small

$$|D_x v^\epsilon| + \tau |D_y v^\epsilon| \leq C \quad \text{in} \quad K \times B_{\frac{r}{2}}(x_0, t_0),$$

under the above conditions in A, B and δ. $\square$

Theorem 3.1.1 establishes that there exist a sequence $\epsilon_j \to 0^+$ such that v^{ϵ_j} converges V locally uniformly in Q_T.

3.1.1 Cell problems

To construct the hamiltonian in the variational inequality (2.1.21) satisfied by the limits of v^ϵ, consider the following cell problems. Due to the unbounded behavior of v^ϵ near $\partial\Omega$, we must consider the cell problems on sets K satisfying $K \subset\subset \Omega$ and supp $u_0 \subset\subset K \times \mathbb{R}^n$. For fixed $x_0, p_0 \in \mathbb{R}^n$, let χ_K^δ a solution of

$$\begin{aligned} F^\delta(D^2\chi, D\chi, y) &= H_K^\delta(x_0, p_0) && \text{in } K \\ \chi &= \infty && \text{on } \partial K \end{aligned} \tag{3.1.6}$$

We also consider for $r > 0$, $\phi(x,t)$ smooth, the solution $\underline{\chi}_{r,K}^\delta$ of

$$\begin{aligned} \underline{F}_r^\delta(D^2\chi, D\chi, y) &= \underline{H}_{r,K}^\delta(x_0, p_0) && \text{in } K \\ \chi &= \infty && \text{on } \partial K. \end{aligned} \tag{3.1.7}$$

and $\overline{\chi}^{\delta}_{r,K}$ of

$$\overline{F}^{\delta}_{r}(D^2\chi, D\chi, y) = \overline{H}^{\delta}_{r,K}(x_0, p_0) \quad \text{in } K \tag{3.1.8}$$
$$\chi = \infty \quad \text{on } \partial K,$$

where

$$p_0 = D\phi(x_0, t_0) \quad \text{and} \quad h(y) = Tr[A(x_0)p_0 \times p_0] + c(x_0, y)$$

and $\underline{F}^{\delta}_{r}$, $\overline{F}^{\delta}_{r}$ are defined as the homogeneous operators in (2.1.9) and (2.1.10). The approximate cell problems to be considered in the case of Dirichlet boundary conditions are listed below.

For $\lambda > 0$, let $\chi^{\lambda,\delta}_{K}$ solve

$$\lambda\chi + F^{\delta}(D^2\chi, D\chi, y) = 0 \quad \text{in} \quad K \tag{3.1.9}$$
$$\chi = \infty \quad \text{on} \quad \partial K.$$

Let $\underline{\chi}^{\lambda,\delta}_{r,K}$ be the solution of

$$\lambda\chi + \underline{F}^{\delta}_{r}(D^2\chi, D\chi, y) = 0 \quad \text{on } K \tag{3.1.10}$$
$$\chi = \infty \quad \text{on } \partial K,$$

and $\overline{\chi}^{\lambda,\delta}_{r,K}$ be the solution of

$$\lambda\chi + \overline{F}^{\delta}_{r}(D^2\chi, D\chi, y) = 0 \quad \text{on } K$$
$$\chi = \infty \quad \text{on } \partial K.$$

For the case of Dirichlet boundary conditions, we need interior gradient bounds to obtain the properties of the cell problem in this case. The next theorem establishes the gradient bounds for the solutions of the cell problem.

Theorem 3.1.2. *For any compact $K \subset \Omega$, there exist a $C_K > 0$ independent of $\lambda > 0$ and $\delta > 0$ such that if $\chi^{\delta,\lambda}$ satisfies*

$$\begin{aligned} \lambda\chi + F^\delta(D^2\chi, D\chi, y) = 0 \quad & in \quad \Omega \\ \chi = \infty \quad & in \quad \partial\Omega \end{aligned}$$

then

$$\|\chi^{\delta,\lambda}\|_{W^{1,\infty}(K)} \leq C_K$$

Proof. We follow the arguments in [9]. Take $\phi \in C_0^\infty$ such that

$$|D^2\phi| \leq C\phi^\theta \qquad |D\phi| \leq C\phi^\theta$$

$$0 \leq \phi \leq 1 \text{ on } \Omega \quad \text{ and } \quad \phi = 1 \text{ on } \Omega_\delta,$$

for some $\theta \in (0,1)$. Let $z = \phi|Dw|^2$, and consider the operator

$$\begin{aligned} L[z] = -\delta Tr[BD^2z] + 2Tr[B(Dw \otimes Dz)] + \frac{2\delta}{\phi}Tr[B(Dz \times D\phi)] + 2\lambda z \\ + 2\delta\phi Tr[BD^2wD^2w]. \end{aligned}$$

At a maximum point y_0 of z we have

$$2\delta\phi Tr[BD^2wD^2w] \leq C|Dw|^2\phi^\theta + C|Dw|^3\phi^{\frac{\theta}{2}} + \mu\delta\phi|D^2w|^2. \tag{3.1.11}$$

Since B is bounded in Ω and $B \geq \mu I$, then

$$\delta Tr[BD^2wD^2w] \geq \frac{C}{\delta}(\delta Tr[BD^2w])^2 \geq \frac{C}{\delta}(\mu|Dw|^2 + C_1)^2, \tag{3.1.12}$$

where C_1 is a lower bound for $\lambda w + h(y)$. Using (3.1.11) and (3.1.12) we obtain

$$\phi|Dw|^4 \leq \delta(C|Dw|^2\phi^\theta + C|Dw|^3\phi^{\frac{\theta}{2}}). \tag{3.1.13}$$

Hence $\max_\Omega \phi w = \phi w(y_0) \leq C$ $\square$

Lemma 3.1.3. The following are the properties of the solution of each cell problem:

a) There exist a unique viscosity solution $\chi_K^{\lambda,\delta}$ satisfying (3.1.9).

b) For $\delta > 0$, $r > 0$, there exist locally a Lipschitz continuous functions $\overline{\chi}_{r,K}^{\delta,\lambda}$ and $\underline{\chi}_{r,K}^{\delta,\lambda}$ satisfying (3.1.10) and (3.1.9) in the viscosity sense.

c) For a point $y_0 \in K$, $\chi_K^{\lambda,\delta} - \chi_K^{\lambda,\delta}(y_0)$ is bounded in $W_{loc}^{1,\infty}(K)$ uniformly in λ and δ. For $y_0 \in K$, $x_0, p_0 \in \mathbb{R}^n$ and fixed δ, $\chi_K^{\lambda,\delta}(y) - \chi_K^{\lambda,\delta}(y_0) \to \chi_K^{\delta}(y)$ locally uniformly in K, and $\lambda\chi_K^{\lambda,\delta} \to H_K^{\delta}(x^0, p^0)$ uniformly in K as $\lambda \to 0^+$.

d) For $r > 0$, consider and a point $y_0 \in K$, $\overline{\chi}_{r,K}^{\lambda,\delta} - \overline{\chi}_{r,K}^{\lambda,\delta}(y_0)$ is bounded in $W_{loc}^{1,\infty}(K)$ uniformly in λ and δ. For $x_0, p_0 \in \mathbb{R}^n$ and fixed δ, $\overline{\chi}_{r,K}^{\lambda,\delta}(y) - \overline{\chi}_{r,K}^{\lambda,\delta}(y_0) \to \overline{\chi}_{r,K}^{\delta}(y)$ locally uniformly in K, and $-\lambda\overline{\chi}_{r,K}^{\lambda,\delta} \to \overline{H}_{r,K}^{\delta}(x_0, p_0)$ uniformly in K as $\lambda \to 0^+$. Similar properties can be proved for $\underline{\chi}_{r,K}^{\lambda,\delta}$, $\underline{\chi}_{r,K}^{\delta}(y)$ and $\underline{H}_{r,K}^{\delta}(x_0, p_0)$.

e) For $\delta > 0$, $\overline{H}_{r,K}^{\delta}(x^0, p^0), \underline{H}_{r,K}^{\delta}(x^0, p^0) \to H_K^{\delta}(x^0, p^0)$ as $r \to 0^+$. There exist a constant $H_K(x^0, p^0)$ such that $H_K^{\delta}(x^0, p^0) \to H_K(x^0, p^0)$ as $\delta \to 0^+$.

f) For each $\gamma > 0$ there exist a $K' \subset\subset \Omega$ such that for each $x, p \in \mathbb{R}^n$, $0 \leq H(x, p) - H_K(x, p) < \gamma$ for all K satisfying $K' \subset\subset K$.

Proof. The existence of $\underline{\chi}_K^{\lambda,\delta}$, $\underline{\chi}_{r,K}^{\lambda,\delta}$ and $\overline{\chi}_{r,K}^{\lambda,\delta}$ follows from Perron's method once a subsolution $\underline{z}$ and a supersolution $\overline{z}$ is constructed. Both $\underline{z}$ and $\overline{z}$ must satisfy

$$\lim_{y\to\partial K} \underline{z}(y) = \infty \tag{3.1.14}$$

$$\lim_{y\to\partial K} \overline{z}(y) = \infty. \tag{3.1.15}$$

Let $A > 0$ and $\gamma > 0$ constants to be chosen later. Define for $\{d(y) < \gamma\}$

$$z_1(y) = -A\delta \log\left[\frac{d(y)}{\gamma}\right] - \frac{2\|h\|}{\lambda}$$

where $d(y)$ =dist$(y, \partial K)$. Then

$$\begin{aligned}&\lambda z_1 - \delta Tr[BD^2\chi] + Tr[BD\chi\otimes D\chi] + h(y) \leq \\ &\qquad \frac{A^2\delta^2}{d(y)^2}\left[\left(1-\frac{1}{A}\right)\mu + \frac{\gamma}{A}|B||(D^2d)^+| + \frac{\gamma^2}{A^2\delta^2}\|h\| + \frac{\lambda\gamma C_\gamma}{\delta A}\right]\end{aligned}$$

for some $C_\gamma > 0$. Take $\gamma = c\delta$, where $c > 0$ will be chosen later, then

$$\begin{aligned}&\lambda z_1 - \delta Tr[BD^2 z-1] + Tr[BDz_1\otimes Dz_1] + h(y) \leq \\ &\qquad \frac{A^2\delta^2}{d(y)^2}\left[\left(1-\frac{1}{A}\right)\mu + \frac{c\delta}{A}|B||(D^2d)^+| + \frac{c^2}{A^2}\|h\| + \frac{\lambda c C_\gamma}{A}\right] \leq 0\end{aligned}$$

in K for δ, λ small and for fixed $0 < A < 1$, choose $c = c(\|h\|, A)$ sufficiently small. To extend z_1 to K, notice that by its construction $z_1(y) < -\frac{\|h\|}{\lambda}$ on $\{d(y) > \gamma\}$. Hence the function

$$\underline{z}(y) = \max\left[z_1(y), -\frac{\|h\|}{\lambda}\right]$$

is a subsolution to (3.1.9) in K and satisfies (3.1.14). To construct $\overline{z}(y)$, consider

$$\overline{z}(y) = -A\delta \log d(y)$$

where A is a constant to be determined later and $d(y)$ satisfies:

- d is a smooth function equal to the distance function to $\partial\Omega$ on $\{\mathrm{dist}(x, \partial K) < \gamma\}$ for some $\gamma > 0$ sufficiently small,
- $0 < d(y) < 1$ on K,
- d satisfies $|Dd| = 1$ on $\{\mathrm{dist}(x, \partial\Omega) < \gamma\}$.

The above properties yield

$$\lambda\overline{z} + F^{\delta}(D^2\overline{z}, D\overline{z}, y) \geq \frac{A^2\delta^2}{d(y)^2}\left[\left(1 - \frac{1}{A}\right)\mu - \frac{|B||(D^2d)^-|}{A}\right] \geq 0$$

for A sufficiently large. On $\{\mathrm{dist}(x, \partial\Omega) \geq \gamma\}$, we have

$$\lambda\overline{z} + F^{\delta}(D^2\overline{z}, D\overline{z}, y) \geq -\frac{A\delta^2}{\gamma^2}|B||(D^2d)^-| + \inf_{\Omega} h(y) \geq 0$$

in K for δ sufficiently small. Then $\overline{z}$ is a supersolution to (3.1.9) in K and satisfies (3.1.15). Thus the existence of $\underline{\chi}_K^{\lambda,\delta}$ is established.

Since all the arguments in the constructions of $\underline{z}$ and $\overline{z}$ are independent of the x dependencies of B and h, then Perron method establishes the existence of $\underline{\chi}_{r,K}^{\lambda,\delta}$. The interior local bounds from the theorem 3.1.2 and analogous arguments in [9] establish $c)$ and $d)$. To prove $e)$, we need to recall that the stability property of viscosity solutions establishes that $\underline{\chi}_{r,K}^{\delta,\lambda}$ and $\overline{\chi}_{r,K}^{\delta,\lambda}$ converge locally uniformly to $\chi_K^{\delta,\lambda}$ in K as $r \to 0$. Then $\underline{H}_{r,K}^{\delta}$ and $\overline{H}_{r,K}^{\delta}$ converge to H_K^{δ} as $r \to 0$. Since $\underline{z} \leq \chi_K^{\delta,\lambda} \leq \overline{z}$ in K, then

$$|\lambda\chi_K^{\delta,\lambda}| \leq 2|h| + C\delta\lambda\log[d(y)]. \tag{3.1.16}$$

Then for every $K' \subset\subset \overline{K}$, there exist a $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ $\|\lambda \chi_K^{\delta,\lambda}\|_{\infty,K'} \leq 3h$. Since h is bounded in Ω then H_K^δ is bounded independently of $K \subset\subset \overline{\Omega}$ and $\delta > 0$. The gradients bounds for the approximate cell problem in the appendix imply that if $H_K^\delta \to H_K$ as δ goes to zero and χ_K^δ converge locally uniformly to χ_K as $\delta \to 0$ in K, then χ_K satisfies

$$Tr[BD\chi \otimes \chi] + h(y) \geq H_K \quad \text{in} \quad K \tag{3.1.17}$$

$$Tr[BD\chi \otimes \chi] + h(y) \leq H_K \quad \text{in} \quad K. \tag{3.1.18}$$

The lack of boundary conditions in the above equation prevents us from applying the uniqueness arguments for the hamiltonian as in Lemma 2.1.3. □

Remark 3.1.1. In the rest of the chapter, $\epsilon = \epsilon_j$ where the sequence ϵ_j is defined by the property $v_j^\epsilon \to V$.

3.1.2 Convergence of the v^ϵ

In this section, the limits of v^ϵ are characterized as the unique viscosity solution of a variational inequality involving the hamiltonian constructed in the previous subsection. The next theorem establishes the equation satisfied by each limit of v^ϵ.

Theorem 3.1.3. *Let v^ϵ to be the solution to (3.1.3). If $v^{\epsilon_j} \to V$, then V is a viscosity solution of*

$$\max[V_t + H(x, DV), V] = 0 \text{ in } \mathbb{R}^n \times (0,T) \tag{3.1.19}$$

$$V(x,0) = \begin{cases} 0 & x \in \overline{Int\{x| \text{ there exist } y \text{ s.t. } \ u_0(x,y) > 0\}} \\ \infty & x \in \overline{\{x| \text{ there exist } y \text{ s.t. } \ u_0(x,y) > 0\}}^c. \end{cases} \tag{3.1.20}$$

Proof. To analyze the small time behavior of v^ϵ, we use the following estimate from Lemma 3.1.1

$$O(\epsilon) \le v^\epsilon \le \min\left[\frac{1}{t}\left[M|x|^2 + \frac{\epsilon}{\delta}\frac{e^{\tau|y|^2}}{d(y)}\right], \frac{1}{r^2-|x|^2} + \frac{\epsilon}{\delta}\frac{1}{r^2-|y|^2}\right] + C\epsilon$$

on $\mathbb{R}^n \times \Omega \times [0,\epsilon]$. Hence, (3.1.20) holds.

First we establish the viscosity subsolution condition for V

$$v_t + H(x, Dv) \le 0 \quad \text{in} \quad \{V > 0\}.$$

Suppose that $V - \phi$ has a local maximum at (x_0, t_0) with $(V-\phi)(x_0,t_0) = 0$ and

$$\phi_t + H(x, D\phi) = \beta > 0 \quad \text{at} \quad (x_0, t_0).$$

Take $K \subset\subset \Omega$ such that

$$|H(x_0,p_0) - H_K(x_0,p_0)| < \frac{\beta}{10}.$$

Choose $\epsilon > 0$ sufficiently small such that

$$\begin{aligned} |H_K^\delta(x_0,p_0) - H_K(x_0,p_0)| &< \frac{\beta}{10} \\ be^{-\frac{1}{\epsilon}V(x_0,t_0)} &< \frac{\beta}{10}. \end{aligned}$$

Take $r > 0$ sufficiently small such that

$$\begin{aligned} \|\phi_t - \phi_t(x_0,t_0)\|_{\infty, B_r(x_0,t_0)} &< \frac{\beta}{10} \\ \|D^2\phi - D^2\phi(x_0,t_0)\|_{\infty, B_r(x_0,t_0)} &< \frac{\beta}{10} \\ |\underline{H}_{r,K}^\delta(x_0,p_0) - H_K^\delta(x_0,p_0)| &< \frac{\beta}{10}. \end{aligned}$$

Define $\phi^\epsilon = \phi(x,t) + \frac{\epsilon}{\delta}\chi(y)$ in $K \times B_r(x_0,t_0)$ where $\chi = \underline{\chi}^\delta_{r,K}$ be the solution of (3.1.7). We claim that ϕ_ϵ is a supersolution of (3.1.3) in $K \times B_r(x_0,t_0)$. Plugging ϕ^ϵ into (3.1.3) we get

$$\begin{aligned}
L[\phi] = \phi_t - \epsilon Tr[AD^2\phi] + F^\delta[\chi] - be^{-\frac{\phi^\epsilon}{\epsilon}} &\geq \phi_t + \underline{F}^\delta_r[\chi] - \frac{3\beta}{10} \\
&\geq \phi_t + \underline{H}^\delta_{r,K}(x_0,p_0) - \frac{6\beta}{10} \\
&\geq \phi_t + H(x_0,p_0) - \frac{7\beta}{10} \\
&= \beta - \frac{7\beta}{10} > 0.
\end{aligned}$$

Since $\phi^\epsilon \to \infty$ as $y \to \partial K$ and v^ϵ is locally bounded independent of ϵ in $K \times B_r(x_0,t_0)$, then there exist $\alpha > 0$ independent of ϵ and K such that

$$v^\epsilon - \phi^\epsilon < -\alpha \quad \text{on} \quad \partial(K \times B_r(x_0,t_0)).$$

Then the maximum principle implies $v^\epsilon - \phi^\epsilon < -\alpha$ on $K \times B_r(x_0,t_0)$. For $y_0 \in$ Int K,

$$0 = (V - \phi)(x_0,t_0) = \overset{*}{\lim_{B_\epsilon(x_0,y_0,t_0)}} [v^\epsilon - \phi^\epsilon] < -\alpha$$

a contradiction. Therefore V satisfies

$$V_t + H(x_0, DV) \leq 0 \quad \text{in} \quad \{V > 0\}$$

in the viscosity sense.

To show that V is a supersolution, suppose that $V - \phi$ has a local minimun at (x_0,t_0) with $(V-\phi)(x_0,t_0) = 0$ and

$$\phi_t + H(x_0, D\phi) = -\beta < 0.$$

Take $\epsilon > 0$ sufficiently small such that

$$|H^\delta(x_0,p_0) - H(x_0,p_0)|_{\infty,B_r(x_0,t_0)} < \frac{\beta}{10}$$

and $r > 0$ sufficiently small such that

$$\begin{aligned}
\|\phi_t - \phi_t(x_0,t_0)\|_{\infty,B_r(x_0,t_0)} &< \frac{\beta}{10} \\
\|D^2\phi - D^2\phi(x_0,t_0)\|_{\infty,B_r(x_0,t_0)} &< \frac{\beta}{10} \\
|\overline{H}_r^\delta(x_0,p_0) - H^\delta(x_0,p_0)| &< \frac{\beta}{10}.
\end{aligned}$$

Define $\phi^\epsilon = \phi + \epsilon\chi$, where $\chi = \overline{\chi}_r^\delta$. We show that ϕ^ϵ is a subsolution to (3.1.3) in $\Omega \times B_r(x_0,t_0)$.

$$\begin{aligned}
L[\phi^\epsilon] &\leq \phi_t + F^\delta[\chi] - \frac{2\beta}{10} \leq \phi_t + \overline{F}_r^\delta[\chi] - \frac{2\beta}{10} \\
&\leq \phi_t + \overline{H}_r^\delta(x_0,p_0) - \frac{2\beta}{10} \\
&\leq \phi_t + H(x_0,p_0) - \frac{4\beta}{10} \leq \phi_t + H(x_0,p_0) - \frac{5\beta}{10} \leq \beta - \frac{5\beta}{10} < 0.
\end{aligned}$$

Since $V - \phi$ have a local minimum at (x_0,t_0), then there exist a constant $\alpha > 0$ such that $V - \phi > 2\alpha$ on $\partial B_r(x_0,t_0)$. Choose compact sets $K' \subset K \subset\subset \Omega$ such that supp $u_0 \subset\subset K' \times R^n$ and

$$v_K^\epsilon - v^\epsilon < \frac{\alpha}{4} \quad \text{in} \quad K' \times B_r(x_0,t_0)$$

where v_K^ϵ is the solution to (3.1.3) in $\mathbb{R}^n \times K \times [0,T]$. Since $v_K^\epsilon \geq v^\epsilon$ on $K \times B_r(x_0,t_0)$ then for ϵ sufficiently small $v^\epsilon - \phi^\epsilon > \alpha$ on $\partial B_r(x_0,t_0) \times K$. This implies $v_K^\epsilon - \phi^\epsilon > \frac{\alpha}{2}$ in $\partial(B_r(x_0,t_0) \times K)$, then the maximum principle implies

$$v_K^\epsilon - \phi^\epsilon > \frac{\alpha}{2} \quad \text{in} \quad B_r(x_0,t_0) \times K.$$

Hence

$$0 = (V - \phi)(x_0, t_0) = \overset{*}{\lim_{\epsilon \to 0}}[v^\epsilon - \phi^\epsilon] = \overset{*}{\lim_{\epsilon \to 0^+}}[v_K^\epsilon - \phi^\epsilon] + \overset{*}{\lim_{\epsilon \to 0^+}}[v^\epsilon - v_K^\epsilon] > \frac{\alpha}{4},$$

a contradiction. Therefore V is a supersolution to (3.1.19). □

3.1.3 Convergence of u^ϵ

Once we have shown the convergence of the functions v^ϵ, we establish the convergence of u^ϵ the solutions of

$$\begin{aligned} &u_t - \epsilon Tr[AD_x^2 u] - \frac{\delta^2}{\epsilon} Tr[BD_y^2 u] + \frac{1}{\epsilon} f(x, y, u) = 0 && \text{in} \quad \mathbb{R}^n \times \Omega \times (0, T) \\ &u(x, y, t) = 0 && \text{in} \quad \mathbb{R}^n \times \partial\Omega \times [0, T] \\ &u(x, y, 0) = u_0(x, y) && \text{in} \quad \mathbb{R}^n \times \Omega \end{aligned}$$

Theorem 3.1.4. *There is $\delta_j = \epsilon_j^\alpha$ where $\alpha \in (0, \frac{1}{2})$, $A = A(x)$ and B a constant matrix then*

$$u^{\epsilon_j} \longrightarrow \begin{cases} 1 & \textit{locally uniformly in} \quad Int\{V = 0\} \times \Omega \\ 0 & \textit{locally uniformly in} \quad Int\{V > 0\} \times \Omega \end{cases} \tag{3.1.21}$$

and if $\delta = \epsilon^\alpha$ with $\alpha \in [\frac{1}{2}, 1)$, $A = A(x)$ and $B = B(x, y)$ then

$$u^{\epsilon_j} \longrightarrow \begin{cases} 1 & \textit{locally uniformly in} \quad Int\{V = 0\} \times \Omega \\ 0 & \textit{locally uniformly in} \quad Int\{V > 0\} \times \Omega \end{cases}$$

where V is the viscosity solution of the equation (3.1.19) with the hamiltonian $H(x, p)$ given by $H(x, p) = \lim j \to \infty H^{\delta(\epsilon_j)}(x, p)$.

Proof.

The same argument as in Theorem 2.1.4 establishes that $u^\epsilon \to 0$ locally uniformly in $\{V > 0\}$. The next step is to show that for any compact subset C of $\mathrm{Int}\{V = 0\} \times \Omega$ we have

$$\liminf_{\epsilon \to 0^+} u^\epsilon > m_C > 0. \tag{3.1.22}$$

Let $(x_0, t_0) \in \mathrm{Int}\{V = 0\}$ and $r > 0$ such that $B_r(x_0, t_0) \subset\subset \mathrm{Int}\{V = 0\}$. Take $\phi(x, t) = |x - x_0|^2 + (t - t_0)^2$, then $V - \phi$ has a local maximum at (x_0, t_0) in $B_r(x_0, t_0)$. Take $K \subset\subset \Omega$ be an open set and let $d_K(y)$ be a smooth function equal to distance$(x, \partial K)$ in a neighborhood of ∂K, such that $0 < d_K(y) < 1$ in K. Define on $K \times B_r(x_0, t_0)$

$$\phi^\epsilon(x, y, t) = \phi(x, t) + \frac{\epsilon}{d_K(y)}.$$

Then let $(x_\epsilon, y_\epsilon, t_\epsilon)$ be the local maximum of $v^\epsilon - \phi^\epsilon$ in $B_r(x_0, t_0) \times K$. Since $\{v^\epsilon\}_{\epsilon > 0}$ is uniformly bounded in $K \times B_r(x_0, t_0)$ and $\phi^\epsilon = \infty$ on $B_r(x_0, t_0) \times \partial K$, then $y_\epsilon \in \mathrm{Int}\ K$. Similarly, the fact that $v^\epsilon \to 0$ uniformly in $K \times B_r(x_0, t_0)$ and $\phi^\epsilon \geq \phi$ implies that $(x_\epsilon, t_\epsilon) \notin \partial B_r(x_0, t_0)$, then

$$\phi_t^\epsilon - \epsilon Tr[AD_x^2\phi^\epsilon] - \frac{\delta^2}{\epsilon} Tr[BD_y^2\phi^\epsilon] + Tr[AD_x\phi^\epsilon \otimes D_x\phi^\epsilon] + \frac{\delta^2}{\epsilon^2} Tr[BD_y\phi^\epsilon \otimes D_y\phi^\epsilon] \leq b\underline{u}^\epsilon - c(x_\epsilon, y_\epsilon).$$

Since $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$ as $\epsilon \to 0^+$, then

$$u^\epsilon(x_\epsilon, y_\epsilon, t_\epsilon) \geq \frac{1}{b}\left[c(x_\epsilon, y_\epsilon) + O(\epsilon) + \frac{\delta^2}{d^2} Tr[BD^2 d] - 2\frac{\delta^2}{d^3} Tr[BDd \otimes Dd] + \frac{\delta^2}{d^4} Tr[BDd \otimes Dd]\right]. \tag{3.1.23}$$

If $y_\epsilon \to \partial\Omega$ as $\epsilon \to 0^+$, then there exist a sufficiently small $\gamma > 0$ such that for sufficiently small ϵ, $y_\epsilon \in \{d_K < \gamma\}$ and (3.1.23) implies

$$u^\epsilon(x_\epsilon, y_\epsilon, t_\epsilon) \geq \inf_{K\times B_r(x_0,t_0)} \frac{1}{b}[c + O(\epsilon)] \geq m > 0$$

for some $m > 0$. If $y_\epsilon \in \{d_K \geq \gamma\}$, then (3.1.23) implies that for ϵ sufficiently small

$$u^\epsilon(x_\epsilon, y_\epsilon, t_\epsilon) \geq \inf_{K\times B_r(x_0,t_0)} \frac{1}{b}[c + O(\epsilon) + O(\delta^2)] \geq m > 0.$$

Hence $u^\epsilon(x_\epsilon, y_\epsilon, t_\epsilon) \geq m > 0$ for sufficiently small ϵ. For $y \in K$

$$(v^\epsilon - \phi^\epsilon)(x_\epsilon, y_\epsilon, t_\epsilon) \geq (v^\epsilon - \phi^\epsilon)(x_0, y, t_0)$$

since $\phi^\epsilon \geq 0$ and $\phi(x_0, t_0) = 0$ then the above implies

$$v^\epsilon(x_\epsilon, y_\epsilon, t_\epsilon) \geq v^\epsilon(x_0, y, t_0) - \frac{\epsilon}{d(y)}.$$

Using the fact that $v^\epsilon = -\epsilon \log u^\epsilon$, we obtain

$$u^\epsilon(x_0, y, t_0) \geq u^\epsilon(x_\epsilon, y_\epsilon, t_\epsilon) e^{\frac{-1}{d(y)}} \geq m e^{\frac{-1}{d(y)}}.$$

Hence for $K' \subset\subset K$

$$\liminf_{\epsilon\to 0^+} u^\epsilon(x_0, y, t_0) \geq m'_K \quad \text{in} \quad K'.$$

This establishes (3.1.22). To show

$$\liminf_{\epsilon\to 0} u^\epsilon = 1 \qquad \text{locally uniformly in} \qquad \mathrm{Int}\{V = 0\} \times \Omega,$$

let $y_0 \in \Omega$ and $r > 0$ such that $B_r(x_0) \times (t_0, t_0 + T) \subset\subset \mathrm{Int}\{V = 0\}$ and $B_r(y_0) \subset\subset \Omega$. For any $\eta \in [\alpha, 1]$, there exist a $\beta = \beta(m, \eta) > 0$ such that

$$f(x, y, u) \leq \beta[u - 1 + \eta]$$

in $(u,x,y) \in [\frac{m}{2}, 1] \times B_r(x_0) \times B_r(y_0)$. Consider the equation

$$w_t - \epsilon Tr[AD_x^2 w] - \frac{\delta^2}{\epsilon} Tr[BD^2 w] + \frac{\beta}{\epsilon}[w - 1 + \eta] = 0 \qquad (3.1.24)$$

$$\text{in} \quad B_r(x_0) \times B_r(y_0) \times (t_0, t_0 + T)$$

$$w = 0 \qquad \text{in} \quad B_r(x_0) \times \partial B_r(y_0) \times (t_0, t_0 + T)$$

$$w = u^\epsilon \qquad \text{in} \quad \partial B_r(x_0) \times B_r(y_0) \times (t_0, t_0 + T).$$

Define for $\lambda > 0$ and $0 < \eta' < 1$

$$\phi_1(y) = \frac{2}{\pi} \tanh[\lambda(r^2 - |y - y_0|^2)] - \eta'.$$

Then $-1 - \eta' \leq \phi \leq 1 - \eta'$ and for sufficiently large λ

$$-Tr[BD^2 \phi_1] \leq 2\lambda \left[\frac{2\lambda R^2 \tanh + \nu n}{\cosh^2} \right] \leq \lambda^2 C_1$$

where $C_1 > 0$ does not depend on λ. Define

$$\varphi(y) = \begin{cases} \varphi_1(y) & \text{in} \quad K \\ 0 & \text{otherwise,} \end{cases}$$

then φ is a viscosity subsolution of

$$-\frac{\delta^2}{\epsilon} Tr[BD^2 \varphi] = \lambda^2 C_1 \frac{\delta^2}{\epsilon} \qquad \text{in} \quad \Omega \qquad (3.1.25)$$

$$\varphi = 0 \qquad \text{in} \quad \partial\Omega.$$

Let

$$z^\epsilon(x,y,t) = f^\epsilon(t)\varphi(y)\phi(x) - \epsilon\gamma(t - t_0)$$

where f^ϵ and ϕ are defined as in Theorem 1.3.1. Next, it is shown that z^ϵ is a viscosity solution of (3.1.24). First, by construction of ϕ and φ,

- z is continuous $B_r(x_0) \times B_r(y_0) \times [t_0, t_0 + T]$.
- $z^\epsilon \leq 0$ on $B_r(x_0) \times \partial B_r(y_0) \times [t_0, t_0 + T]$.
- $z^\epsilon \leq u^\epsilon$ on $\partial B_r(x_0) \times B_r(y_0) \times [t_0, t_0 + T]$.

Let $z^\epsilon - \psi$ have a local maximum at $(x_\epsilon, y_\epsilon, t_\epsilon) \in B_r(x_0) \times B_r(y_0) \times (t_0, t_0 + T)$, then by (3.1.25)

$$\begin{aligned}\psi_t - \epsilon Tr[AD_x^2\psi] - \frac{\delta^2}{\epsilon} Tr[BD_y^2\psi] + \frac{\beta}{\epsilon}[z^\epsilon - 1 + \eta] \leq \frac{\beta - \tau}{\epsilon}[f(t)\varphi(y)\phi(x) + \eta - 1] \\ -\frac{\tau}{\epsilon}(1 - \eta)[1 - \phi\varphi] + \lambda^2 C_1 \frac{\delta^2}{\epsilon} \\ \leq \frac{1}{\epsilon}[-\tau(1 - \eta)\eta' + \lambda^2 C_1 \delta^2] \leq 0\end{aligned}$$

for δ sufficiently small. Then maximum principle implies

$$u^\epsilon \geq w^\epsilon \geq z^\epsilon \qquad \text{in} \qquad B_r(x_0) \times B_r(y_0) \times [t_0, t_0 + T]$$

and

$$\liminf_{\epsilon \to 0} u^\epsilon \geq \liminf_{\epsilon \to 0} z^\epsilon \geq (1 - \eta)(1 - \eta') \qquad \text{in} \quad B_{\frac{r}{2}}(x_0) \times B_r(y_0) \times (t_0, t_0 + T).$$

As η, η' and λ are arbitrary

$$\liminf_{\epsilon \to 0} u^\epsilon \geq 1$$

on $B_{\frac{r}{2}}(x_0) \times B_r(y_0) \times (t_0, t_0 + T)$. The opposite inequality holds since $u^\epsilon \leq 1$ in $\mathbb{R}^n \times \Omega \times [0, T]$, then (3.1.21) holds. $\square$

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